

LINE ARRANGEMENTS AND CONFIGURATIONS OF POINTS WITH AN UNUSUAL GEOMETRIC PROPERTY

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ABSTRACT. The SHGH conjecture proposes a solution to the question of how many conditions a general union of fat points imposes on the complete linear system of curves in \mathbb{P}^2 of fixed degree d , and it is known to be true in many cases. We propose a new problem, namely to understand the number of conditions imposed by a general union of fat points on the incomplete linear system defined by the condition of passing through a given finite set of points Z (not general). Motivated by work of Di Gennaro-Ilardi-Vallès and Faenzi-Vallès, we give a careful analysis for the case where there is a single general fat point, which has multiplicity $d - 1$. There is an expected number of conditions imposed by this fat point, and we study those Z for which this expected value is not achieved. We show, for instance, that if Z is in linear general position then such unexpected curves do not exist. We give criteria for the occurrence of such unexpected curves and describe the range of values of d for which they occur. We also exhibit examples where the unexpected curve is even irreducible. Furthermore, we relate properties of Z to properties of the arrangement of lines dual to the points of Z . In particular, we obtain a new interpretation of the splitting type of a line arrangement. Finally, we use our results to establish a Lefschetz-like criterion for Terao's conjecture on the freeness of line arrangements.

1. INTRODUCTION

A fundamental problem in algebraic geometry is the study of the dimension of linear systems on projective varieties, and many tools have been developed by researchers to this end (e.g. the different versions of the Riemann-Roch theorem). It is usually the case that there is an *expected* dimension (or codimension), given by naively counting constants; understanding the *special* linear systems, i.e., those whose actual dimensions are greater than the expected ones, is a subtle problem of substantial interest.

For example, consider the complete linear system $\mathcal{V} = \mathcal{L}_j$ of plane curves of degree j ; its (projective) dimension is $\binom{j+2}{2} - 1$. For $j \geq m$, the requirement that the curves all have multiplicity at least m at a fixed point P imposes $\binom{m+1}{2}$ linear conditions, and the linear subsystem of all such curves indeed has codimension $\binom{m+1}{2}$ in \mathcal{V} , so the actual and expected codimensions coincide. We will refer to this as the linear subsystem of curves passing through a *fat point of multiplicity m* supported at P . It is a very well-studied (but still open) problem to compute the dimension of the linear subsystem of \mathcal{L}_j of curves of degree j passing through

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a *general* set of r fat points P_1, \dots, P_r with multiplicities m_1, \dots, m_r . The still open *SHGH Conjecture* gives a putative solution to this problem; we will recall this conjecture in more detail below. When $m_1 = \dots = m_r = 2$, results of Alexander-Hirschowitz not only confirm the SHGH Conjecture for those cases, but also solve the corresponding problem for double points in projective spaces of dimension greater than 2; however, little is known for fat points with arbitrary multiplicity in higher dimensions.

Motivated by results in this paper described below, we propose a refinement of the above problem. That is, rather than beginning with $\mathcal{V} = \mathcal{L}_j$, we propose to begin with the linear system $\mathcal{V} = \mathcal{L}_{Z,j}$ of all plane curves of degree j containing a fixed, reduced 0-dimensional scheme Z . We then impose the passage through a general set X of fat points and ask for the dimension of the resulting linear subsystem. The expected dimension depends only on the dimension of the homogeneous component $[I_Z]_j$ of degree j of the ideal of Z and the number of points of X , counted with multiplicity: each point of multiplicity m is expected to impose $\binom{m+1}{2}$ independent conditions, as long as the expected dimension of the linear system is non-negative.

The problem in this generality is currently inaccessible; the case that X is an arbitrary finite general set of fat points and $Z = \emptyset$, for example, has only a conjectural solution, given by the still open SHGH Conjecture. So for this paper we begin a study of this problem by focusing on the first nontrivial case at the other extreme, namely, X a single fat point of multiplicity $j - 1$ and Z an arbitrary finite reduced set of points. It is surprising (as the example of [DIV] in the next paragraph shows) that already in this case, it is no longer true that the expected dimension is necessarily achieved, as it was when we began with $\mathcal{V} = \mathcal{L}_j$ (i.e., when X is one fat point and $Z = \emptyset$). Since Z is *not* assumed to be a general set of points, the problem obtains a new and central aspect, namely to understand how the geometry of Z can affect the desired dimension. In this paper we carefully analyze this surprising behavior. Furthermore, we show that our results have interesting connections to the study of line arrangements. In particular, they give new perspectives on Terao's freeness conjecture, including a generalization to non-free arrangements.

Our original inspiration came in two ways, from a paper of Di Gennaro, Ilardi, and Vallès [DIV]. The first was by an example of [DIV], in which they observe that the set of nine points in \mathbb{P}^2 dual to the so-called B3 arrangement has an unusual geometric property [DIV, Proposition 7.3]: For every point P of the plane, there is a degree four curve passing through these nine points and vanishing to order three at P . This is surprising because a naive dimension count suggests that the linear system of curves of degree 4 containing the nine points and $3P$ should be empty except for a special locus of points P , but in fact it is nonempty for a general point P .

This led us to study finite sets of points Z in the plane for which, for some integer j , the dimension of the linear system of plane curves of degree $j + 1$ that pass through the points of Z and have multiplicity j at a general point P is unexpectedly large. In this case, we say that Z admits an *unexpected curve* of degree $j + 1$ (see Definition 2.1). We establish a numerical criterion for the occurrence of unexpected curves. It involves two invariants. The first one, which arose already in the work of Faenzi and Vallès [FV], we call the *multiplicity index* m_Z of Z . It is the least integer j such that the linear system of degree $j + 1$ forms vanishing at $Z + jP$ is not empty (see Definition 2.5). The second invariant, which is new, is $t_Z := \min \{j \geq 0 : h^0(\mathcal{I}_Z(j + 1)) - \binom{j+1}{2} > 0\}$ (see Definition 2.8). It depends only on the Hilbert function of Z .

It turns out that a set Z of points can have unexpected curves of various degrees. To understand this range of degrees we introduce another new invariant, u_Z , called the *speciality index* of Z , as the least integer j such that the scheme $Z + jP$, where P is a general point, imposes independent conditions on forms of degree $j+1$ (see Definition 2.12). Our first main result (see Theorem 2.16) is:

Theorem 1.1. *Z admits an unexpected curve if and only if $m_Z < t_Z$. Furthermore, in this case Z has an unexpected curve of degree j if and only if $m_Z < j \leq u_Z$.*

In particular, the existence of an unexpected curve forces Z to be *generally special*, that is, $m_Z < u_Z$. The converse, however, is not necessarily true: if Z is generally special, it does not necessarily have an unexpected curve. The introduction and study of generally special sets of points is a key step in establishing our existence criterion for unexpected curves. Another key ingredient is the study of the dimension of the linear system of curves of degree $m_Z + 1$, as Z admits any unexpected curve if and only if this linear system contains an unexpected curve. We show in Theorem 4.1 that the dimension of the linear system of degree $m_Z + 1$ curves is always one, unless $u_Z = m_Z - 1$. In the latter case, the dimension is two. It follows that any unexpected curve of degree $m_Z + 1$ is uniquely determined by Z and the general point P . The mentioned dimension result also allows us to predict how the multiplicity index changes when one adds to Z another general point Q (see Proposition 4.10). Furthermore, we use it to derive a criterion for when the linear system of degree $m_Z + 1$ curves through $Z + m_Z P$ consists of a unique *irreducible* curve (see Corollary 4.11).

Checking for the existence of unexpected curves requires computing m_Z and t_Z . Since t_Z depends only on the fixed reduced scheme Z , it is typically easy to compute. In contrast m_Z is much harder to compute rigorously (although one can get evidence for its value using randomly selected points P). Work of Faenzi and Vallès [FV] relates m_Z to properties of the arrangement of lines \mathcal{A}_Z dual to the points of Z .

Recall that associated to any line arrangement \mathcal{A} is a locally free sheaf \mathcal{D} of rank two, called the derivation bundle. Restricted to a general line L , it splits as $\mathcal{O}_L(-a) \oplus \mathcal{O}_L(-b)$ with $a + b = |Z| - 1$. The pair (a, b) , where $a \leq b$, is called the *splitting type* of \mathcal{D} or \mathcal{A} . For $\mathcal{A} = \mathcal{A}_Z$, let us denote by \mathcal{D}_Z the associated derivation bundle. Then [FV, Theorem 4.3] shows that the number a is equal to the multiplicity index m_Z . We extend this by showing that the multiplicity index m_Z and the speciality index u_Z satisfy the relation $m_Z + u_Z = |Z| - 2$ (see Corollary 4.2), and thus that $b = u_Z + 1$.

This allows us to translate our results about finite sets of points into statements on line arrangements. In the other direction, we use methods for studying line arrangements to determine multiplicity indices of sets of points. For example, we determine the multiplicity index and the speciality index of a set of points in linearly general position and conclude that it does not admit any unexpected curves (see Corollary 5.19). We also show that the set of points dual to the Fermat configurations of $3t \geq 15$ lines admit unexpected curves of degrees $t + 2, \dots, 2t - 3$ (see Proposition 5.13). Furthermore, we exhibit a family of free line arrangements with the property that any of the dual sets of points admits a unique unexpected curve which is in fact irreducible (see Proposition 5.22).

The second way that [DIV] inspired us relates to a fundamental open problem in the study of hyperplane arrangements, namely, Terao's conjecture, which is open even for line arrangements. A line arrangement $\mathcal{A} = \mathcal{A}(f)$ is said to be *free* if the Jacobian ideal of f is saturated, where f is the product of linear forms defining the lines in \mathcal{A} . Terao conjectured

that freeness is a combinatorial property, that is, it depends only on the incidence lattice of the lines in \mathcal{A} . In [DIV], the authors give an equivalent version of Terao's conjecture in terms of Lefschetz properties. In trying to understand their proof we realized that some of the results used in [DIV] to derive the claimed equivalence are not quite true as stated. We use our results on points to clarify and to adjust the needed results. For example, in Theorem 6.5 we show that it is the existence of an unexpected curve (rather than \mathcal{D}_Z being unstable) that is equivalent to the failure of a certain Lefschetz property. We also establish that Terao's conjecture is equivalent to a Lefschetz-like condition (see Proposition 7.6). This allows us to show that the (adjusted) Lefschetz condition given in [DIV] implies Terao's conjecture (see Corollary 7.7). While we do not know if this condition is also necessary, we do establish that Terao's conjecture follows if one can show that the splitting type of a free line arrangement is a combinatorial property (see Corollary 7.4). We wonder, if, in fact, the splitting type of an arbitrary line arrangement is determined by its incidence lattice.

We end the introduction with the more detailed discussion of the SHGH Conjecture which we promised above in the context of the larger problem which frames the work we are doing here. Let $V = [R]_j$ be the vector space of degree j forms in three variables, let \mathcal{L}_j be its projectivization, and let $X = m_1P_1 + \cdots + m_rP_r$ be a fat point scheme supported on a set of r points P_1, \dots, P_r . Thus X is defined by

$$I_X = I_{P_1}^{m_1} \cap \cdots \cap I_{P_r}^{m_r}.$$

We say that X *fails to impose the expected number of conditions* on V (or on \mathcal{L}_j) if

$$\dim_K [I_X]_j > \max \left\{ 0, \dim_K V - \sum_i \binom{m_i + 1}{2} \right\} = \max \left\{ 0, \binom{j + 2}{2} - \sum_i \binom{m_i + 1}{2} \right\}.$$

If the points P_i are general, it is a well-known and difficult open problem to classify all m_i and j such that the subscheme X fails to impose the expected number of conditions on V , but a conjectural answer is given by the SHGH Conjecture [Se, Ha1, G, Hi]. Segre's version of the conjecture, which ostensibly gives only a necessary criterion, is as follows.

Conjecture 1.2 (SHGH Conjecture). *For $X = m_1P_1 + \cdots + m_rP_r$ with general points P_i , X fails to impose the expected number of conditions on V only if $[I_X]_j \neq 0$ but the base locus of $[I_X]_j$ contains a non-reduced curve.*

In fact, the SHGH Conjecture as stated above is equivalent to versions [Ha1, G, Hi] that not only provide an explicit and complete list of all (m_1, \dots, m_r) and j for which $[I_X]_j$ conjecturally fails to impose independent conditions on V but which also conjecturally determine the extent to which the conditions fail to be independent. Although we will not discuss the details here, we note that it took 40 years [CM] to recognize that the partial characterization as given above actually provides a full quantitative conjectural solution.

Similarly, our focus here will be on identifying failures of independence in a generalized context, with a long term goal of obtaining a more complete characterization. The generalized context is that we consider the case that V is a subspace of R_j , in particular, $V = [I_Z]_j$, where Z is a fat point subscheme. Then the overall problem becomes:

Problem 1.3. *Characterize and then classify all triples (Z, X, j) where $Z = c_1Q_1 + \cdots + c_sQ_s$ for distinct points Q_i , $X = m_1P_1 + \cdots + m_rP_r$ for general points P_i , such that X fails to impose the expected number of conditions on $V = [I_Z]_j$.*

If Z is the empty set, then $V = [R]_j$, so this is addressed by the SHGH Conjecture. If Z is reduced, $r = 1$, and $j = m_1 + 1$, this becomes the problem of deciding the existence of an unexpected curve of degree j . Our results give the following answer:

Theorem 1.4. *Let $Z \subset \mathbb{P}^2$ be a finite set of points whose dual is a line arrangement with splitting type (a, b) . Let P be a general point. Then the subscheme $X = mP$ fails to impose the expected number of conditions on $V = [I_Z]_{m+1}$ if and only if*

- (i) $a \leq m \leq b - 2$; and
- (ii) $h^1(\mathcal{I}_Z(t_Z)) = 0$.

Notice that Condition (ii) is equivalent to the assumption $\dim_K[R/I_Z]_{t_Z} = |Z|$.

It would be interesting to understand exactly for which sets Z the above failure of imposing the expected number of conditions occurs. Furthermore, our results strongly suggest that finding answers to Problem 1.3 in other cases is worth investigating.

Our paper is organized as follows: In Section 2 we introduce the basic invariants, establish some preliminary results, and state our criterion for the occurrence of unexpected curves. Section 3 is entirely geared towards establishing this criterion. Our further results on the invariants, dimensions of linear systems, and characterizations of generally special set of points are derived in Section 4. The connection to line arrangements is made in Section 5. In particular, we show Theorem 1.4 there. In Section 6 we introduce the Lefschetz properties and relate them to the existence of unexpected curves. The relation of the Lefschetz properties to Terzio's freeness conjecture is made precise in Section 7.

2. A CRITERION FOR THE OCCURRENCE OF UNEXPECTED CURVES

Throughout, K denotes an infinite field. We denote the homogeneous coordinate ring of \mathbb{P}^2 over K by $R = K[\mathbb{P}^2] = K[x, y, z]$. Given distinct points $P, P_1, \dots, P_s \in \mathbb{P}^2$ and an integer $a > 0$, we denote by $Z = P_1 + \dots + P_s$ the scheme defined by the ideal $I_Z = I_{P_1} \cap \dots \cap I_{P_s} \subseteq R$, where I_{P_i} is the ideal generated by all forms that vanish at P_i , and by $X = Z + jP$ the scheme defined by the ideal $I_X = I_P^j \cap I_Z$. (In particular, Z will always be nonempty.) In each degree t , note that $\dim_K[I_X]_t \geq \dim_K[I_Z]_t - \binom{j+1}{2}$; i.e., the forms in $[I_X]_t$ are obtained from those of $[I_Z]_t$ by imposing at most $\binom{j+1}{2}$ linear conditions coming from jP . Typically, if $\dim_K[I_X]_t > \dim_K[I_Z]_t - \binom{j+1}{2}$ (i.e., if jP imposes fewer than $\binom{j+1}{2}$ conditions on $[I_Z]_t$) for a general point P , it is because $\dim_K[I_Z]_t < \binom{j+1}{2}$ and $\dim_K[I_X]_t = 0$. For special choices of Z , however, it can happen that jP imposes fewer than $\binom{j+1}{2}$ conditions even though P is general and $\dim_K[I_X]_t > 0$. We are interested in exploring this situation when the degree t is $j + 1$. This motivates the following definition, where we denote the sheafification of a homogeneous ideal I by \mathcal{I} . Also, given a sheaf \mathcal{F} on \mathbb{P}^2 , we will usually write $h^0(\mathbb{P}^2, \mathcal{F})$ simply as $h^0(\mathcal{F})$. Thus, for example, $\mathcal{I}_Z \otimes \mathcal{I}_P^j = \mathcal{I}_X$, $h^0(\mathbb{P}^2, \mathcal{I}_Z(t)) = h^0(\mathcal{I}_Z(t)) = \dim_K[I_Z]_t$ and $h^0(\mathbb{P}^2, (\mathcal{I}_Z \otimes \mathcal{I}_P^j)(t)) = h^0((\mathcal{I}_X)(t)) = \dim_K[I_X]_t$.

Definition 2.1. We say that a reduced finite set of points $Z \subset \mathbb{P}^2$ *admits an unexpected curve* if there is an integer $j > 0$ such that, for a general point P , jP fails to impose the expected number of conditions on the linear system of curves of degree $j + 1$ containing Z . That is, Z admits an unexpected curve of degree $j + 1$ if

$$(2.1) \quad h^0((\mathcal{I}_Z \otimes \mathcal{I}_P^j)(j+1)) > \max \left\{ h^0(\mathcal{I}_Z(j+1)) - \binom{j+1}{2}, 0 \right\}.$$

Remark 2.2. While it certainly can be of interest to ask when different kinds of non-reduced schemes admit “unexpected curves” of this sort, in this paper we are concerned only with the case where Z is reduced. Thus we assume from now on that Z is a reduced, zero-dimensional subscheme of \mathbb{P}^2 whose degree is at least two and will not repeat this throughout the paper.

Remark 2.3. If $0 \leq j \leq 1$ and P is general, then $h^0((\mathcal{I}_Z \otimes \mathcal{I}_P^j)(j+1)) > 0$ implies $h^0((\mathcal{I}_Z \otimes \mathcal{I}_P^j)(j+1)) = h^0(\mathcal{I}_Z(j+1)) - \binom{j+1}{2} \geq 0$. Thus unexpected curves must have degree at least 3.

Example 2.4. By Remark 2.3, the least degree for which an unexpected curve can occur is 3. We now show that unexpected curves of degree 3 can occur. Although the occurrence of unexpected curves is not purely a characteristic $p > 0$ phenomenon (later we will give examples in characteristic 0), we believe 2 is the only characteristic for which an unexpected curve of degree 3 can occur. In any case, for this example we assume K to have characteristic 2, and for simplicity we assume K is infinite. For this example, we show that (2.1) holds with $j+1 = 3$ and with the right hand side of (2.1) being 0. Take Z to be the seven points whose homogeneous coordinates $[a : b : c]$ consist of just zeroes and ones. Note that the points are the points of the Fano plane and that any line through two of them goes through a third. There are only seven such lines, and they are projectively dual to the seven points. Let $P = [\alpha : \beta : \gamma] \in \mathbb{P}^2$ be a general point. One can check that Z imposes independent conditions on cubics (in fact, $I_Z = (yz(y+z), xz(x+z), xy(x+y))$). Since $Z+2P$ imposes 10 conditions, one would expect that there would not be a cubic containing Z having a double point at P . But one can easily check that $F = \alpha^2 yz(y+z) + \beta^2 xz(x+z) + \gamma^2 xy(x+y)$ defines a curve C (reduced and irreducible in fact) which is singular at P and hence C is an unexpected curve of degree 3 for Z .

In order for Z to have an unexpected curve of some degree $j+1$, it must be true that $[I_{Z+jP}]_{j+1} \neq 0$. Thus it is worth looking for the least j such that this holds. We will use this quantity to give a criterion for the occurrence of unexpected curves.

Definition 2.5 ([FV, Definition 4.1]). (i) Given a point $P \notin Z$, we call

$$m_{Z,P} = \min\{j \geq 0 : h^0((\mathcal{I}_Z \otimes \mathcal{I}_P^j)(j+1)) > 0\}$$

the *multiplicity index of Z with respect to P* .

(ii) By semicontinuity, fixing Z and j , the integer $h^0((\mathcal{I}_Z \otimes \mathcal{I}_P^j)(j+1))$ is the same for all general points $P \notin Z$ (and its value is the minimum value of $h^0((\mathcal{I}_Z \otimes \mathcal{I}_P^j)(j+1))$ among points $P \notin Z$). Thus, for a general point P ,

$$m_Z = m_{Z,P}$$

is independent of the choice of P and is said to be the *multiplicity index of Z* .

We note that $m_{Z,P}$ exists for each point $P \notin Z$, since it is easy to see that $h^0((\mathcal{I}_Z \otimes \mathcal{I}_P^j)(j+1)) > 0$ holds for all $j \geq \max\{0, |Z|\}$ (pick j lines through P which also go through the $|Z|$ points of Z), and hence $m_{Z,P} \leq \max\{0, |Z|\}$. We also note that $h^0((\mathcal{I}_Z \otimes \mathcal{I}_P^j)(j+1))$ is a nondecreasing function of j , since we have an injection $[I_Z \cap I_P^j]_{j+1} \rightarrow [I_Z \cap I_P^{j+1}]_{j+2}$ given by multiplication by any linear form ℓ vanishing at P . Thus if $h^0((\mathcal{I}_Z \otimes \mathcal{I}_P^j)(j+1)) = 0$, then $m_Z > j$. The following exact sequence makes these observations more precise.

Lemma 2.6. *Let $Z \subset \mathbb{P}^2$ be a finite set of points, and let $P \in \mathbb{P}^2$ be any point that is not in Z . Then, for every integer $j > 0$, multiplication by a general linear form $\ell \in I_P$ on R/I_{Z+jP} induces an exact sequence of graded R -modules*

$$(2.2) \quad 0 \rightarrow (R/I_{Z+(j-1)P})(-1) \rightarrow R/I_{Z+jP} \rightarrow R/(I_{Z+jP}, \ell) \rightarrow 0.$$

Proof. Since P is not in Z by assumption, we get $I_Z : \ell = I_Z$. Moreover, we have $I_P^j : \ell = I_P^{j-1}$. Now the claim follows. \square

As a consequence we see in particular that any finite set Z of points can have at most two independent unexpected curves of degree $m_Z + 1$. More generally, one has.

Lemma 2.7. *Let $Z \subset \mathbb{P}^2$ be a finite set of points, and let $P \in \mathbb{P}^2$ be any point that is not in Z . Then*

$$1 \leq h^0(\mathcal{I}_{Z+m_Z P}(m_Z + 1)) \leq 2.$$

Proof. Let ℓ be a general linear form in I_P . For ease of notation, set $m = m_Z$. The definition of m gives

$$\dim_K[R/I_{Z+(m-1)P}]_m = \binom{m+2}{2} \quad \text{and} \quad \dim_K[R/I_{Z+mP}]_{m+1} < \binom{m+3}{2}.$$

Hence, Sequence (2.2) with $j = m$ yields $\dim_K[R/(I_{Z+mP}, \ell)]_{m+1} \leq m + 1$. Note that the ideal (I_P^m, ℓ) is generated by a regular sequence of two elements with degrees one and m , respectively. Therefore, we obtain the following estimates

$$m = \dim_K[R/(I_P^m, \ell)]_{m+1} \leq \dim_K[R/(I_{Z+mP}, \ell)]_{m+1} \leq m + 1.$$

Using Lemma 2.6, we conclude

$$\binom{m+3}{2} - 2 \leq \dim_K[R/I_{Z+mP}]_{m+1} \leq \binom{m+3}{2} - 1,$$

which is equivalent to the assertion. \square

Our criterion for the existence of unexpected curves will involve an additional quantity; it arises in the right hand side of (2.1). It is also useful in giving an upper bound on m_Z , better than the one above.

Definition 2.8. For a reduced scheme $Z \subset \mathbb{P}^2$ consisting of a finite set of points, set

$$t_Z = \min \left\{ j \geq 0 : h^0(\mathcal{I}_Z(j+1)) - \binom{j+1}{2} > 0 \right\}.$$

To see that t_Z exists, it is helpful to rewrite t_Z using the Hilbert function $h_Z : \mathbb{Z} \rightarrow \mathbb{Z}$ of Z , defined as $h_Z(j) = \dim_K[R/I_Z]_j$.

Lemma 2.9. *For each integer $j \geq 0$ we have:*

- (a) $h^0(\mathcal{I}_Z(j+1)) - \binom{j+1}{2} = 2j + 3 - h_Z(j+1);$
- (b) $h^1(\mathcal{I}_{Z+jP}(j+1)) = h^0(\mathcal{I}_{Z+jP}(j+1)) + |Z| - (2j+3).$

Proof. Part (a) is immediate from the facts that $h^0(\mathcal{I}_Z(j+1)) = \binom{j+3}{2} - h_Z(j+1)$ and $\binom{j+3}{2} - \binom{j+1}{2} = 2j+3$. Then part (b) follows from the exact sequence

$$0 \rightarrow H^0(\mathcal{I}_{Z+jP}(j+1)) \rightarrow H^0(\mathcal{O}_{\mathbb{P}^2}(j+1)) \rightarrow H^0(\mathcal{O}_{Z+jP}(j+1)) \\ \rightarrow H^1(\mathcal{I}_{Z+jP}(j+1)) \rightarrow 0.$$

□

Corollary 2.10. *Let $Z \subset \mathbb{P}^2$ be a reduced scheme consisting of a finite set of points. Then $0 \leq m_Z \leq t_Z \leq \left\lfloor \frac{|Z|-1}{2} \right\rfloor$.*

Moreover, $t_Z = \left\lfloor \frac{|Z|-1}{2} \right\rfloor$ if and only if $h_Z(t_Z) = |Z|$.

Proof. Since the Hilbert function of Z is at most $|Z|$, by Lemma 2.9 we have

$$h^0(\mathcal{I}_Z(j+1)) - \binom{j+1}{2} = 2j+3 - h_Z(j+1) \geq 2j+3 - |Z|.$$

Thus t_Z is always defined and in fact $t_Z \leq \left\lfloor \frac{|Z|-1}{2} \right\rfloor$. Since for any $P \notin Z$ we have $h^0(\mathcal{I}_Z \otimes \mathcal{I}_P^{t_Z})(t_Z+1) \geq h^0(\mathcal{I}_Z(t_Z+1)) - \binom{t_Z+1}{2} > 0$, we see that $0 \leq m_Z \leq m_{Z,P} \leq t_Z$.

The above equality shows that

$$t_Z = \min\{j \geq 0 : h_Z(j+1) \leq 2j+2\}.$$

Now the characterization of the maximum value of t_Z follows. □

Example 2.11. Here we evaluate t_Z exactly when Z lies on a curve of low degree.

- (i) The definition immediately gives that $t_Z = 0$ if and only if the points of Z are collinear, so in this case t_Z is as small as possible, and we also have $m_Z = t_Z$.
- (ii) If Z lies on an irreducible conic, then it is not hard to check that $t_Z = \left\lfloor \frac{|Z|-1}{2} \right\rfloor$, so in this case t_Z is as large as possible. Moreover, we again have $m_Z = t_Z$. (By Bezout's Theorem, any form of degree t_Z vanishing on Z must be divisible by the form defining the conic. But the quotient then has degree $t_Z - 2$, which therefore cannot vanish to order $t_Z - 1$ at a general point P . Thus $m_Z > t_Z - 1$.)

Our criterion involves one more quantity. In order for Z to have an unexpected curve of degree $j+1$, it must be the case for a general point P that $Z + jP$ does not impose independent conditions on forms of degree $j+1$, so the least j for which the conditions are independent is of interest.

Definition 2.12. Let Z be a reduced finite subscheme of \mathbb{P}^2 and let P be a general point. Say that Z is *generally special* if $Z + m_Z P$ does not impose independent conditions on forms of degree $m_Z + 1$, and we define the *speciality index* of Z to be

$$u_Z = \min\{j \geq 0 : h^1(\mathbb{P}^2, \mathcal{I}_{Z+jP}(j+1)) = 0\};$$

i.e., u_Z is the least integer j such that $Z + jP$ imposes independent conditions on forms of degree $j+1$.

We show in Lemma 2.14 that u_Z is well-defined, that is, u_Z is finite. An interpretation of the speciality index in the context of line arrangements in \mathbb{P}^2 will be given in Section 5.

The following observation shows that u_Z can also be characterized as

$$u_Z = \max\{j \in \mathbb{Z} : h^1(\mathbb{P}^2, \mathcal{I}_{Z+jP}(j)) \neq 0\}.$$

Lemma 2.13. *Let $Z \subset \mathbb{P}^2$ be a finite set of points, and let $P \in \mathbb{P}^2$ be any point that is not in Z . Then one has*

$$h^1(\mathbb{P}^2, \mathcal{I}_{Z+jP}(j+1)) = 0 \quad \text{whenever } j \geq u_{Z,P},$$

where $u_{Z,P} = \min\{j \geq 0 : h^1(\mathbb{P}^2, \mathcal{I}_{Z+jP}(j+1)) = 0\}$.

Proof. We use induction on j . If $j = u_{Z,P}$, the claim is true by definition of $u_{Z,P}$. Let $j > u_{Z,P}$. Let $\ell \in I_P$ be a general linear form. Then the saturation of (I_{Z+jP}, ℓ) is $(I_P^j, \ell) = (\ell, f)$ for some form $f \in I_P^j$. Denote by Y the complete intersection defined by this ideal.

Consider now the following part of the cohomology sequence induced by Sequence (2.2)

$$H^1(\mathbb{P}^2, \mathcal{I}_{Z+(j-1)P}(j)) \rightarrow H^1(\mathbb{P}^2, \mathcal{I}_{Z+(P)}(j+1)) \rightarrow H^1(\mathbb{P}^2, \mathcal{I}_Y(j+1)).$$

The induction hypothesis gives that the group on the left-hand side is zero. Since the regularity of Y equals j , the group on the right-hand side vanishes as well. It follows that $H^1(\mathbb{P}^2, \mathcal{I}_{Z+(P)}(j+1)) = 0$, as desired. \square

We now give an explicit upper bound on u_Z .

Lemma 2.14. *Let Z be a set of s points in \mathbb{P}^2 , and let $P \in \mathbb{P}^2$ be a point that is not on any line through two of the points of Z . Then $H^1(\mathbb{P}^2, \mathcal{I}_{Z+(s-2)P}(s-1)) = 0$.*

Proof. We have to show that $Z + (s-2)P$ imposes $s + \binom{s-1}{2}$ conditions to the linear system of plane curves of degree $s-1$. Clearly $(s-2)P$ imposes $\binom{s-1}{2}$ conditions (since the regularity of $(s-2)P$ is $s-1$), so we want to show that the points of Z impose s independent conditions on the linear system, \mathcal{L} , of plane curves of degree $s-1$ vanishing to order $s-2$ at P . It is enough to show that given any point Q of Z there is a curve of degree $s-1$ vanishing to order $s-2$ at the general point P and vanishing at each point of $Z \setminus \{Q\}$, but not vanishing at Q . This can be done (for instance) with a suitable union of $s-1$ lines, each joining P and a point of $Z \setminus \{Q\}$. \square

Corollary 2.15. *Let Z be a reduced finite subscheme of \mathbb{P}^2 .*

- (a) $u_Z \leq |Z| - 2$, with equality if and only if the points of Z lie on a line.
- (b) If Z is generally special, then $m_Z \leq t_Z \leq u_Z$, where one of the inequalities is strict.

Proof. Let $s = |Z|$. If $P \in \mathbb{P}^2$ is a general point, then Lemma 2.14 applies and shows $u_Z \leq |Z| - 2$.

If the points of Z lie on a line ℓ then any curve of degree $\leq s-2$ containing $s-1$ points of Z must also contain the last point, so curves of degree $\leq s-2$ do not separate points of Z , and hence $u_Z = |Z| - 2$.

Conversely, assume that not all points of Z lie on a line. We wish to show that $u_Z \leq s-3$, i.e. that $h^1(\mathcal{I}_{Z+(s-3)P}(s-2)) = 0$. Let Q be any point of Z . We wish to find a curve of degree $s-2$ vanishing to order $s-3$ at P and containing all the points of $Z \setminus \{Q\}$. This will follow from the above construction provided we can show that there is a line ℓ containing two points of $Z \setminus \{Q\}$ but not containing Q . Suppose this is not the case. Then the line joining any two points of $Z \setminus \{Q\}$ must also contain Q . Let A, B be two points of $Z \setminus \{Q\}$ and let ℓ be the line spanned by A and B , so $Q \in \ell$. Let C be any point of $Z \setminus \{A, B, Q\}$.

The line joining C and A also contains Q , so it must be ℓ . In this way we see that all points of Z must lie on ℓ , contradicting our assumption.

Finally, if Z is generally special, then $h^1(\mathbb{P}^2, \mathcal{I}_{Z+m_Z P}(m_Z + 1)) > 0$ by definition, hence $u_Z > m_Z$. But $h^0(\mathbb{P}^2, \mathcal{I}_{Z+m_Z P}(m_Z + 1)) > 0$, so $u_Z > m_Z$ implies

$$h^0(\mathbb{P}^2, \mathcal{I}_{Z+u_Z P}(u_Z + 1)) \geq h^0(\mathbb{P}^2, \mathcal{I}_{Z+m_Z P}(m_Z + 1)) > 0$$

(just add lines through P to get from m_Z to u_Z) with $h^1(\mathbb{P}^2, \mathcal{I}_{Z+u_Z P}(u_Z + 1)) = 0$. Since $Z + u_Z P$ imposes independent conditions on curves of degree $u_Z + 1$, clearly Z alone does as well. Hence

$$\begin{aligned} h^0(\mathbb{P}^2, \mathcal{I}_Z(u_Z + 1)) - \binom{u_Z+1}{2} &= \binom{u_Z+3}{2} - |Z| - \binom{u_Z+1}{2} \\ &= h^0(\mathbb{P}^2, \mathcal{I}_{Z+u_Z P}(u_Z + 1)) \\ &\geq h^0(\mathbb{P}^2, \mathcal{I}_{Z+m_Z P}(m_Z + 1)) \\ &> 0, \end{aligned}$$

proving that $u_Z \geq t_Z$. □

We can now state our criterion for the occurrence of unexpected curves.

Theorem 2.16. *Let Z be a reduced finite subscheme of \mathbb{P}^2 . Then Z has an unexpected curve if and only if $m_Z < t_Z$, in which case $t_Z \leq u_Z$. Moreover, suppose Z has an unexpected curve; then Z has an unexpected curve of degree $j + 1$ if and only if $m_Z \leq j < u_Z$.*

Remark 2.17. (i) In Example 2.4, it turns out that $m_Z = 2$ and $t_Z = 3$, so Theorem 2.16 applies to show that Z has an unexpected curve of degree 3, as we found before by an ad hoc computation.

(ii) Theorem 2.16 also shows that a set of points can admit unexpected curves of different degrees. For a concrete example, see Proposition 5.13, where we exhibit a set of $3t$ points, Z , with $m_Z = t + 1$ and $u_Z = 2t - 3$, and hence for $t \geq 5$ we obtain an unexpected curve in degrees $t + 2, \dots, 2t - 3$.

The proof of Theorem 2.16 requires some preparation, both of which we give in the next section, which also includes some results of interest in their own right. The proof itself can be found immediately after Corollary 3.9.

3. PROOF OF THE CRITERION

If $Z \subset \mathbb{P}^2$ is a finite reduced subscheme with $m_Z < t_Z$, then it is not difficult to see that Z has an unexpected curve of degree $j + 1$ for each $m_Z \leq j < t_Z$. The key for proving Theorem 2.16 is to understand the value of $h_Z(t_Z)$.

Proposition 3.1. *Let $Z \subset \mathbb{P}^2$ be a reduced scheme consisting of a finite set of points. Then conditions (a) and (b) are equivalent:*

- (a) $h_Z(t_Z) < |Z|$;
- (b) (i) the scheme Z is a complete intersection cut out by two curves meeting transversely, of degree 2 and $t_Z + 1$ respectively, with $t_Z > 0$; or
- (ii) there is a line that contains precisely $|Z| - t_Z \geq t_Z + 2$ points of Z .

Furthermore, in case (b)(i) we have $t_Z = \frac{|Z|-2}{2}$, while for case (b)(ii) we have $t_Z \leq \frac{|Z|-2}{2}$.

Proof. To simplify notation, put $t = t_Z$. We use Δh_Z to denote the first difference of the Hilbert function of Z ; that is, $\Delta h_Z(j) = h_Z(j) - h_Z(j-1)$.

First assume $t = 0$. By Remark 2.11, the points of Z are collinear, so (a) holds if and only if $|Z| > 1$, and (b) holds if and only if $|Z| \geq 2$, so (a) and (b) are equivalent, and clearly $0 = t \leq \frac{|Z|-2}{2}$ for $|Z| > 1$. Thus it is now enough to consider the case that $t \geq 1$, that is, that Z is not collinear.

Assume (a) holds. By the definition of t_Z , Lemma 2.9 and the fact that h_Z is strictly increasing until it stabilizes at the value $|Z|$, this forces

$$2t + 1 \leq h_Z(t) < h_Z(t+1) \leq 2t + 2,$$

and thus $h_Z(t+1) = 2t + 2 = 1 + h_Z(t)$. In particular, $\Delta h_Z(t+1) = 1$. By standard results (see, for example, [DGM, Proposition 3.9]), this implies that the values of Δh_Z are as follows (where s is the regularity of I_Z):

$$\begin{array}{cccccccc} j & : & 0 & 1 & \dots & t+1 & \dots & s-1 & s \\ \Delta h_Z(j) & : & 1 & 2 & \dots & 1 & \dots & 1 & 0 \end{array}$$

Thus, $h_Z(t+1) = 2t + 2$ implies

$$|Z| = 2t + 2 + (s - t - 2) = s + t.$$

Using $h_Z(t+1) = 2t + 2 \leq |Z|$, we conclude that $s = |Z| - t \geq t + 2$.

Now we consider two cases:

Case 1: Assume Z does not lie on a conic, that is, $\Delta h_Z(2) = 3$. Hence

$$2t + 2 = h_Z(t+1) = \sum_{j=0}^{t+1} \Delta h_Z(j)$$

forces $\Delta h_Z(t) = 1$, and thus $\Delta h_Z(s-2) = \Delta h_Z(s-1) = 1 > \Delta h_Z(s)$. By [Da, (2.3)] (or by applying results of [BGM]), it follows that $[I_Z]_{s-1}$ has a linear form ℓ as a common divisor. Since I_Z has a minimal set of homogeneous generators all of whose degrees are at most s , there must be a generator f of degree s and by [Ca, Theorem 2.1] there is only one generator of degree s in a minimal set of homogeneous generators. Moreover, since Z is reduced, the curves defined by f and ℓ must intersect transversely. Thus the ideal (ℓ, f) defines a subset of s collinear points of Z and clearly ℓ vanishes at no point of Z other than these s . Therefore, condition (ii) is satisfied.

Case 2: Assume Z is contained in a conic, defined, say, by a homogeneous form q . Again taking into account $h_Z(t+1) = 2t + 2$, we get

$$(3.1) \quad \Delta h_Z(j) = \begin{cases} 1 & \text{if } j = 0 \text{ or } t+1 \leq j < s \\ 2 & \text{if } 1 \leq j \leq t \\ 0 & \text{otherwise.} \end{cases}$$

It follows that q is a common factor for $[I_Z]_j$ for $j \leq t$, but not for $j = t+1$, so any minimal set of homogeneous generators for I_Z must contain q and a generator g of degree $t+1$. If q and g are coprime, then $|Z| \leq \deg(q) \deg(g) = 2t + 2$. Since $|Z| = t + s$ and $s \geq t + 2$, this means $s = t + 2$ and $|Z| = 2t + 2$, so Z is a complete intersection as claimed in (i). Otherwise, q and g have a linear common factor ℓ and I_Z has another minimal generator f of degree s . As in Case 1 we conclude that the line defined by ℓ contains precisely s points of Z , and so condition (ii) is met.

Conversely, assume one of the conditions in (b) is true. Thus, $|Z| - t \geq t + 2$ (and hence $t \leq \frac{|Z|-2}{2}$), by hypothesis for part (ii) and using the fact that Z is a transverse complete intersection of a conic with a curve of degree $t + 1$ for part (i). Again, we consider two cases:

If (i) is true, then $h_Z(t) = 2t + 1 < 2t + 2 = |Z|$, as desired. Moreover, here we have $t = \frac{|Z|-2}{2}$.

Finally, assume (ii) is true, let $Y \subset Z$ be a subset of $|Z| - t$ collinear points and let U be the complement of Y in Z . Then $t = |U|$ and U is reduced, so U imposes independent conditions on forms of degree $t - 1$; i.e., $h_U(t - 1) = t$ and thus $\dim_K[I_U]_{t-1} = \binom{t+1}{2} - t$. But the linear form ℓ vanishing on Y is, by Bezout's Theorem, a common divisor of $[I_Z]_t$, so $\dim_K[I_Z]_t = \dim_K[I_U]_{t-1}$, and we have $h_Z(t) = 2t + 1 < 2t + 2 \leq |Z|$. \square

Remark 3.2. In part (b) (ii), the stated inequality is not automatic and is in fact a crucial assumption. For example, consider a set of points Z for which $\Delta h_Z(j)$ is the sequence $(1, 2, 3, 2, 2, 1)$ and assume that exactly six points lie on a line. One immediately computes that $|Z| = 11$, and a routine calculation gives $t_Z = 5$. Thus it is true that precisely $|Z| - t_Z = 6$ points lie on a line, but it is not true that $6 = |Z| - t_Z \geq t_Z + 2 = 7$. So this example does not satisfy the entire hypothesis of (ii). And indeed, the assertion of (a) is not satisfied: $h_Z(t_Z) = h_Z(5) = 11$ is not strictly less than $|Z|$.

Remark 3.3. Notice that for $j = 0$, the condition $h^0((\mathcal{I}_Z \otimes \mathcal{I}_P^j)(j+1)) > 0$ does not depend on P . Thus, the following conditions are equivalent:

- (a) $m_{Z,P} = 0$ for some point $P \notin Z$.
- (b) $m_Z = 0$.
- (c) The points in Z are collinear.
- (d) $u_Z = |Z| - 2$ (see Corollary 2.15).
- (e) $t_Z = 0$.

Comparing with Example 2.11, we get for a set Z of collinear points $m_Z = t_Z$. See also Corollary 4.4 of [FV].

As we gather the results we need to prove Theorem 2.16, we will also exhibit further cases where $m_Z = t_Z$. We write $Z + Q$ to denote the reduced scheme whose points consist of the points of Z together with Q , where $Q \notin Z$ is another point, and $Z + jP$ the scheme defined by $I_Z \cap I_P^j$ for a point $P \notin Z$.

Lemma 3.4. *Let $Z \subset \mathbb{P}^2$ be a reduced scheme consisting of a finite set of points, and assume the base field K is infinite. Then, for each general point $P \in \mathbb{P}^2$,*

$$h^0((\mathcal{I}_Z \otimes \mathcal{I}_P^j)(j)) > 0 \text{ if and only if } j \geq |Z|.$$

Furthermore, $h^0((\mathcal{I}_Z \otimes \mathcal{I}_P^j)(j)) = h^0((\mathcal{I}_Z)(j)) - \binom{j+1}{2} = j + 1 - |Z|$ and $h^1((\mathcal{I}_Z)(j-1)) = h^1((\mathcal{I}_Z \otimes \mathcal{I}_P^j)(j)) = 0$ for $j \geq |Z|$.

Proof. A form of degree j with multiplicity j at a point P is a product of j linear forms corresponding to a set of j lines concurrent at P . If K is infinite and P is general, no line through two distinct points of Z passes through P , so there is a set of j lines concurrent at P which vanish on Z if and only if $j \geq |Z|$, and the lines are uniquely determined if $j = |Z|$. Thus $h^0((\mathcal{I}_Z \otimes \mathcal{I}_P^j)(j)) = 0$ if $j < |Z|$ and $h^0((\mathcal{I}_Z \otimes \mathcal{I}_P^{|Z|})(|Z|)) = 1$. In particular, by adding suitable lines through P we obtain the first assertion.

Now, $1 = h^0((\mathcal{I}_Z \otimes \mathcal{I}_P^{|Z|})(|Z|)) \geq h^0((\mathcal{I}_Z)(|Z|)) - \binom{|Z|+1}{2} \geq \binom{|Z|+2}{2} - |Z| - \binom{|Z|+1}{2} = 1$, hence $Z + |Z|P$ (and thus Z) imposes independent conditions on forms of degree $|Z|$. This means $h^1((\mathcal{I}_Z)(j)) = 0$ for $j = |Z|$ (and hence for $j \geq |Z|$), and it means $h^1((\mathcal{I}_Z \otimes \mathcal{I}_P^j)(j)) = 0$ for $j = |Z|$. Replacing Z by $Z + Q$ for any point $Q \notin Z$, we now get $h^1((\mathcal{I}_{Z+Q} \otimes \mathcal{I}_P^j)(j)) = 0$ for $j = |Z + Q| = |Z| + 1$ and hence $Z + Q + jP$ imposes independent conditions on forms of degree $j = |Z| + 1$, and therefore $Z + jP$ also imposes independent conditions on forms of degree $j = |Z| + 1$. Continuing in this way, we see that for any $j \geq |Z|$, $Z + jP$ imposes independent conditions on forms of degree j ; hence for such j we have $h^1((\mathcal{I}_Z \otimes \mathcal{I}_P^j)(j)) = 0$. Thus $h^0((\mathcal{I}_Z \otimes \mathcal{I}_P^j)(j)) = \binom{j+2}{2} - |Z| - \binom{j+1}{2} = j + 1 - |Z|$ as asserted. Since $h^1((\mathcal{I}_Z)(j)) = 0$ for $j \geq |Z|$, we also have $h^0((\mathcal{I}_Z)(j)) = \binom{j+2}{2} - |Z|$, so $h^0((\mathcal{I}_Z \otimes \mathcal{I}_P^j)(j))$ can in addition be written as $h^0((\mathcal{I}_Z)(j)) - \binom{j+1}{2}$.

Since we have already shown that $h^1((\mathcal{I}_Z \otimes \mathcal{I}_P^j)(j)) = 0$ for $j \geq |Z|$, it remains only to prove that $h^1((\mathcal{I}_Z)(j-1)) = 0$ for $j \geq |Z|$. But this is true for any finite set of points, so we are done. \square

We are ready to show:

Theorem 3.5. *Let $Z \subset \mathbb{P}^2$ be a reduced scheme consisting of a finite set of points such that $h_Z(t_Z) < |Z|$, assuming the base field K is infinite. Then*

$$m_Z = t_Z < \frac{|Z| - 1}{2}$$

and $h^0((\mathcal{I}_Z \otimes \mathcal{I}_P^{m_Z})(m_Z + 1)) = 1$. Furthermore,

$$h^0((\mathcal{I}_Z \otimes \mathcal{I}_P^j)(j + 1)) = h^0((\mathcal{I}_Z)(j + 1)) - \binom{j + 1}{2}$$

for all $j \geq m_Z$ (hence Z admits no unexpected curves).

Proof. By Proposition 3.1 we have to consider two cases.

Case 1: Assume Z is defined by an ideal $I_Z = (q, g)$, where $q, g \in R$ are forms of degree 2 and $t_Z + 1$, respectively (so $|Z| = 2t_Z + 2$). Then for $j + 1 < t_Z + 1$ we get,

$$[I_Z \cap I_P^j]_{j+1} = [(q) \cap I_P^j]_{j+1} = q \cdot [I_P^j]_{j-1} = 0,$$

which implies $m_Z = t_Z$. Since the dimension of $[I_Z \cap I_P^{m_Z}]_{m_Z+1} = [(q, g) \cap I_P^{m_Z}]_{m_Z+1}$ is at most one more than $\dim_K[(q) \cap I_P^{m_Z}]_{m_Z+1} = 0$, we conclude $h^0(\mathcal{I}_Z \otimes \mathcal{I}_P^{m_Z})(m_Z + 1) = 1$, as desired. And since $|Z| = 2t_Z + 2$, we have $t_Z < \frac{|Z|-1}{2}$.

To show $h^0((\mathcal{I}_Z \otimes \mathcal{I}_P^j)(j + 1)) = h^0((\mathcal{I}_Z)(j + 1)) - \binom{j+1}{2}$ for $j \geq m_Z$, we use the known Hilbert function of Z (see (3.1) for Δh_Z). For $j \geq m_Z$ we have $h_Z(j + 1) = |Z|$, so Z imposes independent conditions on forms of degree $j + 1$. But

$$\begin{aligned} 1 &= h^0((\mathcal{I}_Z \otimes \mathcal{I}_P^{m_Z})(m_Z + 1)) \\ &\geq h^0((\mathcal{I}_Z)(m_Z + 1)) - \binom{m_Z+1}{2} \\ &= \binom{t_Z+3}{2} - (2t_Z + 2) - \binom{t_Z+1}{2} \\ &= 1, \end{aligned}$$

hence $Z + m_Z P$ also imposes independent conditions on forms of degree $m_Z + 1$. This also means that the points of Z impose independent conditions on $H^0((\mathcal{I}_P^{m_Z})(m_Z + 1))$. By adding lines through P , it is then clear that the points of Z also impose independent conditions on

$H^0((\mathcal{I}_P^{m_Z+k})(m_Z+1+k))$, which implies $h^0((\mathcal{I}_Z \otimes \mathcal{I}_P^j)(j+1)) = h^0((\mathcal{I}_Z)(j+1)) - \binom{j+1}{2}$ for all $j \geq m_Z$ as desired.

Case 2: Assume a line defined by a linear form $\ell \in R$ contains precisely $|Z| - t_Z \geq t_Z + 2$ points of Z (and hence $t_Z < \frac{|Z|-1}{2}$) and let Y be the set of these points. Let $U \subset Z$ be the subset of the other t_Z points. Then $I_Y = (\ell, f)$ for some form f , where $\deg f \geq t_Z + 2$. Thus, for each integer j and any general point $P \in \mathbb{P}^2$, we get

$$[I_Z \cap I_P^j]_{j+1} = [(\ell, f) \cap I_U \cap I_P^j]_{j+1}.$$

Since $\deg f \geq t_Z + 2$, it follows for $j+1 \leq t_Z + 1$ that

$$[I_Z \cap I_P^j]_{j+1} = [(\ell) \cap I_U \cap I_P^j]_{j+1} = \ell \cdot [I_U \cap I_P^j]_j$$

because ℓ does not vanish at P or at any of the points in U . Since $|U| = t_Z$, Lemma 3.4 gives $h^0((\mathcal{I}_U \otimes \mathcal{I}_P^j)(j)) \leq 1$ for $j \leq t_Z$, with equality exactly when $j = t_Z$. Thus $m_Z = t_Z$ and $h^0((\mathcal{I}_Z \otimes \mathcal{I}_P^{m_Z})(m_Z+1)) = 1$.

To finish, we have as in [CHT, (1.2.3)] exact sequences of sheaves

$$0 \rightarrow (\mathcal{I}_U)(j) \rightarrow (\mathcal{I}_Z)(j+1) \rightarrow \mathcal{I}_{Y,L}(j+1) \rightarrow 0$$

and

$$0 \rightarrow (\mathcal{I}_U \otimes \mathcal{I}_P^j)(j) \rightarrow (\mathcal{I}_Z \otimes \mathcal{I}_P^j)(j+1) \rightarrow \mathcal{I}_{Y,L}(j+1) \rightarrow 0,$$

where L is the line defined by ℓ and $\mathcal{I}_{Z \cap L, L}$ is the ideal sheaf of $Y = Z \cap L$ regarded as a subscheme of L , and extended by 0 to \mathbb{P}^2 . For $j \geq m_Z = |U|$ we have

$$h^1((\mathcal{I}_U)(j)) = h^1((\mathcal{I}_U \otimes \mathcal{I}_P^j)(j)) = 0$$

by Lemma 3.4, hence the sequences are exact on global sections. Since $L \cong \mathbb{P}^1$, we have $\mathcal{I}_{Y,L}(j+1) \cong \mathcal{O}_{\mathbb{P}^1}(j+1-|Y|)$, so

$$h^0(\mathcal{I}_{Y,L}(j+1)) = 0 \text{ for } j+1-|Y| \leq -1$$

and

$$h^1(\mathcal{I}_{Y,L}(j+1)) = 0 \text{ for } j+1-|Y| \geq -1.$$

Thus the first sequence gives

$$h^0((\mathcal{I}_Z)(j+1)) = h^0((\mathcal{I}_U)(j)) \text{ for } j+1 \leq |Y|-1,$$

and

$$h^0((\mathcal{I}_Z)(j+1)) = h^0((\mathcal{I}_U)(j)) + j+2-|Y| \text{ for } j+1 \geq |Y|-1.$$

And similarly the second sequence gives

$$h^0((\mathcal{I}_Z \otimes \mathcal{I}_P^j)(j+1)) = h^0((\mathcal{I}_U \otimes \mathcal{I}_P^j)(j)) \text{ for } j+1 \leq |Y|-1,$$

and

$$h^0((\mathcal{I}_Z \otimes \mathcal{I}_P^j)(j+1)) = h^0((\mathcal{I}_U \otimes \mathcal{I}_P^j)(j)) + j+2-|Y| \text{ for } j+1 \geq |Y|-1.$$

So for $m_Z < j+1 \leq |Y|-1$, we have

$$\begin{aligned} h^0((\mathcal{I}_Z \otimes \mathcal{I}_P^j)(j+1)) &= h^0((\mathcal{I}_U \otimes \mathcal{I}_P^j)(j)) \\ &= h^0((\mathcal{I}_U)(j)) - \binom{j+1}{2} \\ &= h^0((\mathcal{I}_Z)(j+1)) - \binom{j+1}{2}, \end{aligned}$$

while for $j+1 \geq |Y|-1$, we have $h^1((\mathcal{I}_Z \otimes \mathcal{I}_P^j)(j+1)) = 0$, since

$$h^1((\mathcal{I}_U \otimes \mathcal{I}_P^j)(j)) = 0 = h^1(\mathcal{I}_{Y,L}(j+1)),$$

and hence

$$h^0((\mathcal{I}_Z \otimes \mathcal{I}_P^j)(j+1)) = h^0((\mathcal{I}_Z)(j+1)) - \binom{j+1}{2},$$

completing the proof. \square

Lemma 3.6. *If Z has an unexpected curve, then Z is generally special.*

Proof. If Z has a unexpected curve, then for some j we have that

$$h^0((\mathcal{I}_Z \otimes \mathcal{I}_P^j)(j+1)) > \max \left\{ h^0(\mathcal{I}_Z(j+1)) - \binom{j+1}{2}, 0 \right\}.$$

Thus $h^0((\mathcal{I}_Z \otimes \mathcal{I}_P^j)(j+1)) > 0$ so $j \geq m_Z$, and

$$h^0((\mathcal{I}_Z \otimes \mathcal{I}_P^j)(j+1)) > h^0(\mathcal{I}_Z(j+1)) - \binom{j+1}{2} \geq \binom{j+3}{2} - \binom{j+1}{2} - |Z|,$$

so $h^1((\mathcal{I}_Z \otimes \mathcal{I}_P^{m_Z})(m_Z+1)) \geq h^1(\mathcal{I}_Z \otimes \mathcal{I}_P^j)(j+1) > 0$, hence $Z + m_Z P$ does not impose independent conditions on forms of degree $m_Z + 1$. \square

Proposition 3.7. *Assume that Z is generally special. Then*

- (a) $(m_Z + 1)^2 - m_Z^2 - |Z| \leq -2$.
- (b) $|Z| \geq 2m_Z + 3$.
- (c) $\dim_K [I_{Z+m_Z P}]_{m_Z+1} = 1$ for a general point P .
- (d) If C_Z is the unique curve given by (c), then C_Z has a unique irreducible component, B_Z , whose order of vanishing at P is $\deg(B_Z) - 1$. Every other component of C_Z , if any, is a line joining P and a point of Z not on B_Z .

Proof. Parts (a) and (b) are clearly equivalent, but we will use both formulations so we record both as part of the statement. Moreover, (c) implies the former conditions. Indeed, using (c), Lemma 2.9 and the definition of Z being generally special (so $h^1(\mathcal{I}_{Z+m_Z P}(m_Z+1)) > 0$), we obtain

$$|Z| - (2m_Z + 3) = h^1(\mathcal{I}_{Z+m_Z P}(m_Z+1)) - 1 \geq 0$$

as claimed.

It remains to show (c) and (d). Since P is general, we may assume that P is not on any line through two points of Z , that $m_{Z,P} = m_Z$, and that $Z + m_Z P$ does not impose independent conditions on forms of degree $m_Z + 1$.

If $m_Z = 0$, then the points of Z are collinear, and so there is a unique curve of degree 1 containing $Z + m_Z P = Z$, as claimed. Moreover, in this case $|Z|$ must be at least 3 in order that Z not impose independent conditions on forms of degree 1, hence $(m_Z + 1)^2 - m_Z^2 - |Z| = 1 - |Z| \leq 1 - 3 = -2$.

Hereafter we may assume $m_Z > 0$. Let $r = |Z|$. Let $\pi : X \rightarrow \mathbb{P}^2$ be the morphism obtained by blowing up the plane at P and at the r points p_1, \dots, p_r of Z . Let E_i be the exceptional curve for p_i and let E_P be the exceptional curve for P under this morphism. Let L be the pullback to X of a line in \mathbb{P}^2 .

Let C be a curve of degree $m_Z + 1$ containing $Z + m_Z P$. First consider the case that C is irreducible. Then by the genus formula C is rational, with multiplicity exactly m_Z at P , and is smooth elsewhere. In particular, we claim that there are no infinitely near singular

points. Indeed, let D be the proper transform of C to X . The assertion is that D is smooth. The arithmetic genus, p , of D satisfies

$$2p - 2 = D^2 + K_X \cdot D = ((m_Z + 1)^2 - m_Z^2) - 3(m_Z + 1) + m_Z = -2,$$

hence $p = 0$, so D is smooth and rational, and $p = g$, the geometric genus. Note that D is linearly equivalent to $(m_Z + 1)L - m_Z E_P - E_1 - \cdots - E_r$, so we have $h^1(X, \mathcal{O}_X(D)) = h^1((\mathcal{I}_Z \otimes \mathcal{I}_P^{m_Z})(m_Z + 1)) > 0$ since Z is generally special. Taking cohomology of

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X(D) \rightarrow \mathcal{O}_D(D) \rightarrow 0$$

and using $h^1(X, \mathcal{O}_X) = 0$ and $h^1(X, \mathcal{O}_X(D)) > 0$, we see that $h^1(D, \mathcal{O}_D(D)) > 0$. Since D is a smooth rational curve, we find that $D^2 < -1$, hence $h^0(D, \mathcal{O}_D(D)) = 0$ and so $\dim_K[I_{Z+m_Z P}]_{m_Z+1} = h^0(D, \mathcal{O}_X(D)) = 1$. In this case C is the curve corresponding to the unique irreducible factor of H of some degree t vanishing to order exactly $t - 1$ at P , and we see that $t = \deg(C) = m_Z + 1$.

Now assume that C is not irreducible. Since C has degree $m_Z + 1$ and multiplicity m_Z at P , the sum of the irreducible components of C not through P has degree at most 1. Indeed, if not then the remaining components of C would be a curve of degree less than m_Z with a point of multiplicity m_Z at P . Hence there is at most one component not through P . We consider the two possibilities.

• Case 1: Suppose first that there is one component not through P (call it B). Then B has degree 1, and all other components together comprise a curve of degree m_Z with multiplicity m_Z at P . Thus all of these other components are lines through P ; call them L_1, \dots, L_{m_Z} .

Suppose that B and some L_i meet at a point of Z or that some L_i does not meet Z . Then we can replace L_i by a general line through P , and the result still is a curve containing $Z + m_Z P$ and hence defined by an element of $[I_{Z+m_Z P}]_{m_Z+1}$. But then subtracting L_i from C gives a curve coming from $[I_{Z+(m_Z-1)P}]_{m_Z}$, hence $[I_{Z+(m_Z-1)P}]_{m_Z} \neq 0$, contradicting the definition of m_Z .

So we conclude from this that each L_i joins P to a point P_i . By the generality of P we get that the L_i are distinct and that no L_i passes through two of the points P_i . Thus each L_j joins P to a distinct point of Z , and B contains the remaining points of Z (if any).

• Case 2: If all of the components of C contain P , we first claim that without loss of generality we can assume that they are not all lines. Suppose they were. Since by the generality of P each line through P can contain at most one point of Z , so we would have $m_Z + 1 \geq |Z| = r$ lines as components of C . Let L_i be the line joining P and $p_i \in Z$. By replacing one of the L_i by a general line through P_i , we are left with an element of $[I_{Z+m_Z P}]_{m_Z+1}$ having a component not through P , so we are back in the first case.

Thus one of the components of C , call it A , must have degree $a \geq 2$. The multiplicity of A at P is at most $a - 1$, since A is irreducible. If its multiplicity at P were less than $a - 1$, then the other components together would have degree $m_Z + 1 - a$ but multiplicity at P of $m_Z - (a - 2) > m_Z + 1 - a$, which is impossible. Thus A has multiplicity $a - 1$ at P . Hence A is a rational curve whose proper transform is smooth, and all of the other components of C are lines through P and distinct points of Z . As was the case with B , and for the same reason, none of these lines and A meet at a point of Z . This concludes Case 2.

Now we combine the conclusions of the two cases above. Note that both B and A have proper transforms which are smooth rational curves, and their degrees are more than their multiplicities at P . So we now combine both cases and let B denote a component of C of

degree b with a point of multiplicity $b - 1$ at P , and whose proper transform is a smooth rational curve. And the other components of C are lines through P through some s distinct points of Z , with B containing the remaining $r - s$ points and no other points of Z , and $b + s = m_Z + 1$. Thus once we finish showing that $\dim_K[I_{Z+m_Z P}]_{m_Z+1} = 1$, we will see that B is the curve corresponding to the unique irreducible factor of H of some degree t vanishing to order exactly $t - 1$ at P , and we see that $t = \deg(B)$.

Let F be the proper transform of B , let D_1, \dots, D_s be the proper transforms of the linear components of C , and let $\Sigma_i = D_1 + \dots + D_i$. Since each D_i meets F once, is disjoint from the other D_j , and has $D_i^2 = -1$, induction starting with

$$h^1(X, \mathcal{O}_X) = 0 = h^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) = h^1(D_1, \mathcal{O}_{D_1}(D_1))$$

and $h^0(X, \mathcal{O}_X) = 1$ and

$$0 = h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) = h^0(D_1, \mathcal{O}_{D_1}(D_1))$$

applied to

$$0 \rightarrow \mathcal{O}_X(\Sigma_i) \rightarrow \mathcal{O}_X(\Sigma_{i+1}) \rightarrow \mathcal{O}_{D_{i+1}}(\Sigma_{i+1}) \rightarrow 0$$

gives $h^1(X, \mathcal{O}_X(\Sigma_s)) = 0$ and $h^0(X, \mathcal{O}_X(\Sigma_s)) = 1$. Now from

$$(3.2) \quad 0 \rightarrow \mathcal{O}_X(\Sigma_s) \rightarrow \mathcal{O}_X(F + \Sigma_s) \rightarrow \mathcal{O}_F(F + \Sigma_s) \rightarrow 0,$$

we obtain

$$\mathcal{O}_F(F + \Sigma_s) \cong \mathcal{O}_{\mathbb{P}^1}(F^2 + s) = \mathcal{O}_{\mathbb{P}^1}(2b + 1 + 2s - r)$$

and

$$h^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2b + 1 + 2s - r)) = h^1(F, \mathcal{O}_F(F + \Sigma_s)) = h^1((\mathcal{I}_Z \otimes \mathcal{I}_P^{m_Z})(m_Z + 1)) > 0,$$

so we get $2b + 1 + 2s - r < -1$, hence $h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2b + 1 + s - r)) = 0$, so

$$1 = h^0(X, \mathcal{O}_X(\Sigma_s)) = h^0(X, \mathcal{O}_X(F + \Sigma_s)) = h^0((\mathcal{I}_Z \otimes \mathcal{I}_P^{m_Z})(m_Z + 1)),$$

as desired. \square

Remark 3.8. The upshot of Proposition 3.7 is that if Z is generally special then there is a unique curve of degree $m_Z + 1$ passing through Z and vanishing at a general point with multiplicity m_Z . This uniqueness is not necessarily true if Z is not generally special. For example, if Z is a set of 7 general points then $m_Z = 3$ and $\dim_K[I_{Z+3P}]_4 = 2$.

The mentioned uniqueness is the key for the following consequence.

Corollary 3.9. *Let Z be a reduced finite subscheme of \mathbb{P}^2 . If $h_Z(t_Z) = |Z|$ and Z is generally special, then $m_Z < t_Z \leq u_Z$ (and hence Z has an unexpected curve).*

Proof. If $m_Z = t_Z$, then we have

$$1 = \dim_K[I_{Z+m_Z P}]_{m_Z+1} > \binom{m_Z + 3}{2} - |Z| - \binom{m_Z + 1}{2} = \dim_K[I_Z]_{m_Z+1} - \binom{m_Z + 1}{2} > 0,$$

where: the first equality is by Proposition 3.7 since Z is generally special, the first strict inequality is by definition since Z is generally special, the second equality is because $h_Z(t_Z) = |Z|$, and the last inequality is by definition of t_Z . Since everything is an integer, this is impossible. Thus $m_Z < t_Z$, and we have $t_Z \leq u_Z$ by Corollary 2.15 (b). \square

We are now ready to prove the result promised in the last section.

Proof of Theorem 2.16. If Z has an unexpected curve, then Z is generally special by Lemma 3.6, and $h_Z(t_Z) = |Z|$ by Theorem 3.5. Then Corollary 3.9 gives $m_Z < t_Z$ (in particular). Conversely, if $m_Z < t_Z$ then from the definitions we see that Z has an unexpected curve. The fact that then $m_Z < t_Z \leq u_Z$ follows by Corollary 3.9.

Now assume that Z has an unexpected curve, so $h_Z(t_Z) = |Z|$ and $m_Z < t_Z \leq u_Z$. If Z has an unexpected curve of degree $j + 1$, then $m_Z \leq j$ (since there must be a curve) and $j < u_Z$ by Lemma 2.13 (since to be unexpected $Z + jP$ must not impose independent conditions on forms of degree $j + 1$).

Conversely, assume that $m_Z \leq j < u_Z$. If $j < t_Z$ then Z has an unexpected curve of degree $j + 1$ as noted above. So say $t_Z \leq j < u_Z$. Since $h_Z(t_Z) = |Z|$, we also have $h_Z(j) = |Z|$. Then by definition of m_Z and adding lines through P if necessary we have $h^0(\mathcal{I}_{Z+jP}(j+1)) > 0$ (so there is a curve), and by definition of u_Z , $h^1(\mathcal{I}_{Z+jP}(j+1)) \neq 0$ (so $Z + jP$ fails to impose independent conditions on forms of degree $j + 1$). Thus $h^0(\mathcal{I}_{Z+jP}(j+1)) > \binom{j+3}{2} - |Z| - \binom{j+1}{2} = h^0(\mathcal{I}_Z(j+1)) - \binom{j+1}{2}$, which shows Z has an unexpected curve of degree $j + 1$. \square

We close this section with an example of an occurrence of an unexpected curve. We will return to this example in the context of the vector bundle methods of a subsequent section.

Example 3.10. For this example we assume our ground field has characteristic 0. Consider the line configuration given by the lines defined by the following 19 linear forms: $x, y, z, x + y, x - y, 2x + y, 2x - y, x + z, x - z, y + z, y - z, x + 2z, x - 2z, y + 2z, y - 2z, x - y + z, x - y - z, x - y + 2z, x - y - 2z$, shown in Figure 1. Let Z be the corresponding reduced scheme consisting of the 19 points dual to the lines, sketched in Figure 2.

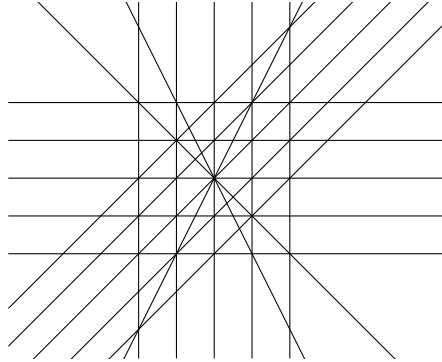


FIGURE 1. A configuration of 19 lines (the line at infinity, $z = 0$, is not shown).

It is not hard to verify that the first difference of the Hilbert function of Z is $\Delta h_Z = (1, 2, 3, 4, 4, 4, 1)$, from which we find that $t_Z = 9$. Picking a random point P , Macaulay2 [M2] finds that $[I_{Z+7P}]_8 = 0$. By upper semicontinuity, this means $m_Z > 7$. Thus we have $8 \leq m_Z \leq t_Z = 9$. Using our results up to now, we cannot determine whether $m_Z = 8$ or $m_Z = 9$. In Example 5.11 we will show that $m_Z = 8$, so Z has an unexpected curve of degree 9.

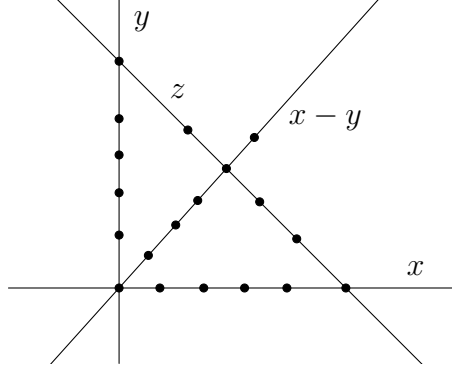


FIGURE 2. A sketch of the points dual to the lines of the line configuration given in Figure 1.

4. FURTHER CRITERIA

In this section we establish additional facts about the invariants that we introduced in the preceding sections, and about the property of being generally special. We also discuss irreducibility of unexpected curves.

Recall that, for a set of points Z of \mathbb{P}^2 and another point $P \notin Z$, the dimension of $H^0(\mathcal{I}_{Z+m_Z P}(m_Z + 1))$ is one or two (see Lemma 2.7). Extending Proposition 3.7, our first result describe which value occurs if P is a general point. It also gives a numerical characterization of a generally special set of points. Notice that the definitions immediately imply that a finite set of points Z is generally special if and only if $m_Z < u_Z$. We continue to use the assumption that Z consists of at least two points.

Theorem 4.1. *Let $Z \subset \mathbb{P}^2$ be a finite set, and let $P \in \mathbb{P}^2$ be a general point. If the base field K is infinite, then one has:*

(a) *The following three conditions are equivalent:*

- (i) $m_Z > u_Z$.
- (ii) $\dim_K[I_{Z+m_Z P}]_{m_Z+1} = 2$.
- (iii) $|Z| = 2m_Z + 1$.

Furthermore, any of these conditions implies $u_Z = m_Z - 1$.

(b) $m_Z = u_Z$ *if and only if* $|Z| = 2m_Z + 2$.

(c) $m_Z < u_Z$ *if and only if* $|Z| \geq 2m_Z + 3$.

Proof. Let ℓ be a general linear form in I_P . As observed above, then $(I_P^{m_Z}, \ell) = (\ell, f)$, where $f \in I_P^{m_Z}$ is a form of degree m_Z . We denote by $H_{\mathfrak{m}}^i(M)$ the i -th local cohomology module of a graded R -module M with support in $\mathfrak{m} = (x, y, z)$.

By definition, $m_Z = 0$ if and only if Z is collinear. In this case, we get $u_Z = |Z| - 2$ by Corollary 2.15. Moreover, $|Z| \geq 2$ by assumption, Hence, we may assume $m_Z \geq 1$ for the remainder of the proof.

First, we show (a). Assume $\dim_K[I_{Z+m_Z P}]_{m_Z+1} = 2$. Then Sequence (2.2) gives

$$m_Z = \dim_K[R/(I_{Z+m_Z P}, \ell)]_{m_Z+1} = \dim_K[R/(\ell, f)]_{m_Z+1}.$$

Since the saturation of $(I_{Z+m_Z P}, \ell)$ is $(I_P^{m_Z}, \ell) = (\ell, f)$, and the latter ideal has minimal generators whose degrees are at most m_Z , we conclude that

$$[(I_{Z+m_Z P}, \ell)]_j = [(\ell, f)]_j \quad \text{whenever } j \geq m_Z + 1.$$

Furthermore, Lemma 3.4 yields $[I_{Z+m_Z P}]_{m_Z} = 0$ because $m_Z \leq t_Z \leq \frac{|Z|-1}{2}$ by Corollary 2.10. It follows that

$$H_{\mathfrak{m}}^0(R/(I_{Z+m_Z P}, \ell)) \cong (\ell, f)/(I_{Z+m_Z P}, \ell) \cong K(-m_Z).$$

The sequence in local cohomology induced by Sequence (2.2) begins

$$0 \rightarrow K(-m_Z) \rightarrow H_{\mathfrak{m}}^1(R/I_{Z+(m_Z-1)P})(-1) \rightarrow H_{\mathfrak{m}}^1(R/I_{Z+m_Z P}) \rightarrow H_{\mathfrak{m}}^1(R/(\ell, f)).$$

By Proposition 3.7, the assumption $\dim_K[I_{Z+m_Z P}]_{m_Z+1} = 2$ implies that Z is not generally special, and hence $u_Z \leq m_Z$. Assume $u_Z = m_Z$. By definition of u_Z , this means that $[H_{\mathfrak{m}}^1(R/I_{Z+m_Z P})]_{m_Z+1} = 0$ and $[H_{\mathfrak{m}}^1(R/I_{Z+(m_Z-1)P})]_{m_Z} \neq 0$, which is a contradiction by the above exact sequence. It follows that $u_Z < m_Z$.

Conversely, assume $u_Z < m_Z$. Consider the beginning of the long exact cohomology induced by Sequence (2.2)

$$0 \rightarrow H_{\mathfrak{m}}^0(R/(I_{Z+m_Z P}, \ell)) \rightarrow H_{\mathfrak{m}}^1(R/I_{Z+(m_Z-1)P})(-1) \rightarrow H_{\mathfrak{m}}^1(R/I_{Z+m_Z P}).$$

By Lemma 2.13 we know $[H_{\mathfrak{m}}^1(R/I_{Z+(m_Z-1)P})]_{m_Z} = 0$. It follows that

$$0 = [H_{\mathfrak{m}}^0(R/(I_{Z+m_Z P}, \ell))]_{m_Z+1} \cong [(\ell, f)/(I_{Z+m_Z P}, \ell)]_{m_Z+1},$$

which implies $\dim_L[(I_{Z+m_Z P}, \ell)/(\ell)]_{m_Z+1} = 2$, and hence $\dim_L[I_{Z+m_Z P}]_{m_Z+1} \geq 2$. Now Lemma 2.7 gives equality. This concludes the proof that conditions (i) and (ii) are equivalent.

By definition of m_Z , we have $h^0(\mathbb{P}^2, \mathcal{I}_{Z+(m_Z-1)P}(m_Z)) = 0$. Applying Lemma 2.9(b) with $j = m_Z - 1$, we get

$$h^1(\mathbb{P}^2, \mathcal{I}_{Z+(m_Z-1)P}(m_Z)) = |Z| - [2(m_Z - 1) + 3] = |Z| - [2m_Z + 1].$$

It follows that $|Z| = 2m_Z + 1$ if and only if $h^1(\mathbb{P}^2, \mathcal{I}_{Z+(m_Z-1)P}(m_Z)) = 0$. By Lemma 2.13, the latter condition is equivalent to $u_Z < m_Z$, which proves that conditions (i) and (iii) are equivalent.

For (a), it remains to show that $u_Z = m_Z - 1$. This is clear if $m_Z = 1$. Let $m_Z \geq 2$. The definition of m_Z also gives $h^0(\mathbb{P}^2, \mathcal{I}_{Z+(m_Z-2)P}(m_Z - 1)) = 0$. Applying Lemma 2.9(b) with $j = m_Z - 2$, we obtain

$$h^1(\mathbb{P}^2, \mathcal{I}_{Z+(m_Z-2)P}(m_Z - 1)) = |Z| - [2(m_Z - 2) + 3] = 2,$$

which proves $u_Z = m_Z - 1$. Now the proof of (a) is complete.

Second, we show (b). Assume $u_Z = m_Z$. Combining part (a) and Lemma 2.7, this gives $\dim_K[I_{Z+m_Z P}]_{m_Z+1} = 1$. Moreover, by definition of u_Z we have $[H_{\mathfrak{m}}^1(R/I_{Z+m_Z P})]_{m_Z+1} = 0$. Hence, applying Lemma 2.9(b) with $j = m_Z$, we obtain $|Z| = 2m_Z + 2$, as desired.

Conversely, assume $|Z| = 2m_Z + 2$. Then part (a) implies $u_Z \geq m_Z$. If $m_Z < u_Z$, then Proposition 3.7 shows $|Z| \geq 2m_Z + 3$. It follows that we must have $u_Z = m_Z$, which completes the proof of (b).

Part (c) follows from (a) and (b). \square

Combining this with earlier results, we see that the multiplicity index and the speciality index are closely related.

Corollary 4.2. *If $Z \subset \mathbb{P}^2$ is any finite set of points, then $m_Z + u_Z = |Z| - 2$.*

Proof. If $2m_Z + 2 \geq |Z|$, then the claim follows from Theorem 4.1(a), (b). Thus, it remains to consider the case, where $2m_Z + 3 \leq |Z|$, that is, Z is generally special. By Proposition 3.7, then the irreducible components of C_Z consist of an irreducible curve B_Z of some degree $t + 1$ with multiplicity t at P and containing exactly $|Z| - r$ points of Z , where $r = m_Z - t$,

and r lines L_i which go through P and the r points of Z not on B_Z . We denote by E the sum of the exceptional curves of the blowings up of the points of Z , by E_P the blow up of P and by H the pullback of a line. Let C_Z^* be the proper transform of C_Z , B_Z^* the proper transform of B_Z and L_i^* the proper transforms of the L_i . Setting $j = m_Z + s$, we have

$$\begin{aligned} h^1(\mathcal{I}_{Z+jP}(j+1)) &= h^1(\mathcal{I}_{Z+(m_Z+s)P}(m_Z+s+1)) \\ &= h^1(\mathcal{O}_X(m_Z+s+1)H - E - (m_Z+s)E_P) \\ &= h^1(\mathcal{O}_X(C_Z^* + s(H - E_P))), \end{aligned}$$

so u_Z is the least value of $s + m_Z$ such that these are 0. Since Z is generally special we know that $u_Z > m_Z$, and thus $s > 0$. Note that $\mathcal{O}_X(C_Z^* + s(H - E_P)) = \mathcal{O}_X(B_Z^* + \sum_i L_i^* + s(H - E_P))$. Thus from the standard short exact sheaf sequence

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X(B_Z^*) \rightarrow \mathcal{O}_{B_Z^*}(B_Z^*) \rightarrow 0,$$

we have

$$0 \rightarrow \mathcal{O}_X\left(\sum_i L_i^* + s(H - E_P)\right) \rightarrow \mathcal{O}_X(C_Z^* + s(H - E_P)) \rightarrow \mathcal{O}_{B_Z^*}(C_Z^* + s(H - E_P)) \rightarrow 0.$$

It is easy to see that h^1 of the leftmost term is 0 (note that the curves L_i^* and $H - E_P$ are disjoint smooth rational curves of self-intersection -1 or 0), so the h^1 of the middle term is at most equal to the h^1 of the rightmost term. But B_Z^* is smooth and rational, so h^1 here is 0 if and only if $(B_Z^*) \cdot (C_Z^* + s(H - E_P)) \geq -1$. Since $(B_Z^*) \cdot (C_Z^* + s(H - E_P)) = (t+1)(m_Z+1) - tm_Z - (|Z| - r) + s = m_Z + t + 1 + r + s - |Z| = 2m_Z + 1 + s - |Z|$, we see being greater than or equal to -1 is equivalent to $s + m_Z \geq |Z| - m_Z - 2$, hence $u_Z = |Z| - m_Z - 2$. \square

We know that the existence of an unexpected curve to a set Z implies that Z is generally special (see Lemma 3.6). The following result shows that the converse is not necessarily true.

Proposition 4.3. *Let $Z \subset \mathbb{P}^2$ be a set of d points such that there is a line L that contains precisely $t \geq \frac{d+2}{2}$ of the points of Z . Then one has:*

- (a) $m_Z = d - t$ and $u_Z = t - 2$.
- (b) Z does not admit an unexpected curve.
- (c) If $t \geq \frac{d+3}{2}$, then Z is generally special.

Proof. Let P be a general point, and set $m = d - t$. Note that $t \geq \frac{d+2}{2}$ implies $t \geq m + 2$. Thus, if $j \leq m$, then L is a fixed component of the linear systems of curves corresponding to both $[I_Z \cap I_P^j]_{j+1}$ and $[I_Z]_{j+1}$. It follows that $\dim_K[I_Z \cap I_P^j]_{j+1} = \dim_K[I_Y \cap I_P^j]_j$ and $\dim_K[I_Z]_{j+1} = \dim_K[I_Y]_j$, where Y consists of the m points of Z not on L . By Lemma 3.4, $\dim_K[I_Y \cap I_P^m]_m = 1$ and $\dim_K[I_Y \cap I_P^{m-1}]_{m-1} = 0$. Thus $\dim_K[I_Z \cap I_P^m]_{m+1} = 1$ but $\dim_K[I_Z \cap I_P^{m-1}]_m = 0$, which gives $m_Z = m = d - t$, and so $u_Z = t - 2$ by Corollary 4.2, proving (a).

Using Lemma 3.4 again, we get $1 = \dim_K[I_Z \cap I_P^m]_{m+1} = \dim_K[I_Y \cap I_P^m]_m = \dim_K[I_Y]_m - \binom{m+1}{2} = \dim_K[I_Z]_{m+1} - \binom{m+1}{2}$, and so $t_Z \leq m = m_Z \leq t_Z$. But now we have $m_Z = t_Z$, and so by Theorem 2.16, Z does not admit an unexpected curve, proving (b).

Finally (c) follows because $|Z| \geq \frac{d+3}{2}$ if and only if $d - t < t - 2$, which by (a) is equivalent to Z being generally special. \square

We now show that adding an irreducibility hypothesis forces the existence of an unexpected curve.

Corollary 4.4. *Let $Z \subset \mathbb{P}^2$ be a generally special set of points such that $[I_{Z+m_Z P}]_{m_Z+1}$ contains an irreducible form and $m_Z \geq 1$. Then Z has an unexpected curve.*

Proof. If $h_Z(t_Z) = |Z|$ then by Corollary 3.9 we obtain that Z has an unexpected curve. (We did not need irreducibility.) Thus, it remains to rule out that $h_Z(t_Z) < |Z|$. Indeed, if $h_Z(t_Z) < |Z|$, then Theorem 3.5 gives $m_Z = t_Z$. By Proposition 3.1, there are two cases.

- In case (b)(i) we have that Z is the complete intersection of a conic and a curve of degree $t_Z + 1$, so $|Z| = 2t_Z + 2$. Hence $|Z| = 2m_Z + 2 < 2m_Z + 3$. Contradiction. (Again we did not use irreducibility.)
- In case (b)(ii), there is a line through $|Z| - t_Z \geq t_Z + 2$ of the points, i.e. through $|Z| - m_Z \geq m_Z + 2$ of the points. Thus this line is a component of any curve of degree $m_Z + 1$ containing $Z + m_Z P$. Since $m_Z > 0$ and $h^0(\mathcal{I}_{Z+m_Z P}(m_Z + 1)) = 1$ by Theorem 3.5, we see that there is no irreducible form in $[I_{Z+m_Z P}]_{m_Z+1}$, contrary to assumption.

□

We summarize some of our results by giving an extension of our criterion for unexpected curves, Theorem 2.16.

Theorem 4.5. *The following conditions are equivalent for a reduced finite subscheme $Z \subset \mathbb{P}^2$.*

- (a) Z has an unexpected curve;
- (b) $m_Z < t_Z$;
- (c) $m_Z < t_Z = \left\lfloor \frac{|Z|-1}{2} \right\rfloor$;
- (d) Z is generally special and $h_Z(t_Z) = |Z|$;
- (e) $h_Z(t_Z) = |Z| \geq 2m_Z + 3$.

Proof. The equivalence of (a) and (b) is Theorem 2.16.

By Theorem 3.5, $m_Z < t_Z$ implies $h_Z(t_Z) = |Z|$, which is equivalent to $t_Z = \left\lfloor \frac{|Z|-1}{2} \right\rfloor$ (see Corollary 2.10). This proves the equivalence of (b) and (c).

Lemma 3.6 shows that (a) implies (d). The fact that (c) implies (a) is Corollary 3.9. The equivalence of (d) and (e) comes from Theorem 4.1. □

As a consequence, we get a first answer to Problem 1.3 in the case, where Z is reduced and $r = 1$.

Proposition 4.6. *Given a finite set of points Z and a general point P . Then the subscheme $X = mP$ fails to impose the expected number of conditions on $V = [I_Z]_{m+1}$ if and only if*

- (i) $h^0(\mathcal{I}_{Z+mP}(m+1)) \cdot h^1(\mathcal{I}_{Z+mP}(m+1)) \neq 0$; and
- (ii) $h^1(\mathcal{I}_Z(t_Z)) = 0$.

Proof. Note that $h^0(\mathcal{I}_{Z+mZ}(m+1)) \neq 0$ is equivalent to $m_Z \leq m$. Furthermore, using Lemma 2.13, one sees that $h^1(\mathcal{I}_{Z+mZ}(m+1)) \neq 0$ is equivalent to $m+1 \leq u_Z$. Hence, Condition (i) of the statement means $m_Z < m \leq u_Z$, which in particular shows that Z is generally special. Condition (ii) is equivalent to $h_Z(t_Z) = |Z|$. We conclude by using Theorems 4.5 and 2.16. □

Remark 4.7. Fix an integer $m \geq 1$, and let $Z \subset \mathbb{P}^2$ be a finite set of points. If P is a general point of \mathbb{P}^2 , we have seen in Lemma 3.4 that mP imposes the expected number of conditions on $[I_Z]_j$ if $j \leq m$, whereas this may fail if $j = m + 1$. We wonder if this degree is the only degree where such failure can occur. In other words, we pose the following

Question: If mP does not impose the expected number of conditions on $[I_Z]_j$ for some integer j , is it then true that j must equal $m + 1$?

We now consider the change of the multiplicity index if one adds a point to a given set of points.

Lemma 4.8. *Let P_1, \dots, P_s, P, Q be distinct points of \mathbb{P}^2 , with P a general point, and let $Z = P_1 + \dots + P_s$. Then $m_{Z+Q,P} = m_Z$ if either $\dim_K[I_Z \cap I_P^{m_Z}]_{m_Z+1} > 1$ or Q is a base point of $[I_Z \cap I_P^{m_Z}]_{m_Z+1}$. Otherwise, $m_{Z+Q,P} = m_Z + 1$.*

Proof. For each integer $j \geq 0$, one has $I_{Z+Q} \cap I_P^j \subset I_Z \cap I_P^j$, hence $m_{Z+Q,P} \geq m_Z$. If Q is a base point of $[I_Z \cap I_P^{m_Z}]_{m_Z+1}$, then $\dim_K[I_{Z+Q} \cap I_P^{m_Z}]_{m_Z+1} = \dim_K[I_Z \cap I_P^{m_Z}]_{m_Z+1} \geq 1$, so $m_{Z+Q,P} = m_Z$. If $\dim_K[I_Z \cap I_P^{m_Z}]_{m_Z+1} > 1$, then, since $\dim_K[I_{Z+Q} \cap I_P^{m_Z}]_{m_Z+1}$ drops by at most 1, we have $\dim_K[I_{Z+Q} \cap I_P^{m_Z}]_{m_Z+1} \geq 1$, and again $m_{Z+Q,P} = m_Z$. If Q is not a base point of $[I_Z \cap I_P^{m_Z}]_{m_Z+1}$, then the dimension drops exactly one, so if also $\dim_K[I_Z \cap I_P^{m_Z}]_{m_Z+1} = 1$ we get $\dim_K[I_{Z+Q} \cap I_P^{m_Z}]_{m_Z+1} = 0$, hence $m_{Z+Q,P} \geq m_Z + 1$. But let $f \neq 0$ be a form of degree $m_Z + 1$ in $I_Z \cap I_P^{m_Z}$. Let ℓ be a linear form that defines the line through P and Q . Then $\ell f \neq 0$ is in $[I_{Z+Q} \cap I_P^{m_Z+1}]_{m_Z+2}$, which shows $m_{Z+Q,P} \leq m_Z + 1$. \square

Thus, given m_Z , there are only two possible values of m_{Z+Q} . When the number points of Z is odd and m_Z is as large as possible, we can say which of these values occurs for an arbitrary point Q .

Proposition 4.9. *Let Z be a finite reduced subscheme of \mathbb{P}^2 . If $m_Z = \frac{|Z|-1}{2}$, then $m_{Z+Q} = m_Z$ for any point Q not in Z .*

Proof. By Corollary 2.10 we get $m_Z = t_Z = \frac{|Z|-1}{2}$. Thus, Lemma 2.9(b) implies

$$h^0((\mathcal{I}_Z \otimes \mathcal{I}_P^{m_Z})(m_Z + 1)) \geq 2m_Z + 3 - |Z| = 2.$$

Now the result follows by Lemma 4.8. \square

If $m_Z < \frac{|Z|-1}{2}$ and Q is a general point, we now find the value of m_{Z+Q} .

Proposition 4.10. *Let Z be a finite reduced subscheme of \mathbb{P}^2 and let Q be a general point. If $m_Z < \frac{|Z|-1}{2}$, then $m_{Z+Q} = m_Z + 1$.*

Proof. If $m_Z < \frac{|Z|-1}{2}$, then Theorem 4.1(a) gives $\dim_K[I_{Z+m_Z P}]_{m_Z+1} = 1$. Hence the result follows from Lemma 4.8. \square

We will observe on more than one occasion below that it is of interest to know when Z admits an *irreducible* unexpected curve of minimal degree $m_Z + 1$. Furthermore, Corollary 4.4 shows that this is related to the existence of an irreducible form in $[I_{Z+m_Z P}]_{m_Z+1}$. These facts motivate the next result.

Corollary 4.11. *Assume that Z is a finite set of points in \mathbb{P}^2 such that $2m_Z + 2 \leq |Z|$. Let $P \in \mathbb{P}^2$ be a general point. Then $[I_{Z+m_Z P}]_{m_Z+1}$ contains an irreducible form if and only if $m_{Z-Q} = m_Z$ for each point $Q \in Z$.*

Proof. By Theorem 4.1, there is a unique curve C of degree $m_Z + 1$ vanishing at Z and to order m_Z at the general point P .

If $m_{Z-Q} = m_Z - 1$ (the only other possibility), let F be a form in $[I_{Z-Q}]_{m_Z}$ vanishing to order $m_Z - 1$ at P ; by abuse of notation we will also denote by F the curve that it defines. Then if ℓ is the line joining Q to P , ℓF must equal C , hence C is not irreducible.

Conversely, assume that $m_{Z-Q} = m_Z$ for all $Q \in Z$. By Proposition 3.7 (iv), if C is not irreducible then there is at least one component of C consisting of a line joining P and a point $Q \in Z$. Removing this point and this line shows that $m_{Z-Q} = m_Z - 1$, giving a contradiction. \square

We will apply this irreducibility criterion to give examples of irreducible unexpected curves in the following section when we have further methods to compute multiplicity indices. We conclude this section by showing that an unexpected curve cannot be completely decomposable, that is, be a union of lines. But first we need a lemma.

Lemma 4.12. *Let Y be a set of m points in \mathbb{P}^2 , and let $P \in \mathbb{P}^2$ be a point that is not on any line through two of the points of Y . Let $X \subset \mathbb{P}^2$ be a set of $u \geq m + 2$ points on a line L such that none of the points in Y nor P is on L . Set $Z = Y \cup X$. Then for $m + 1 \leq j + 1 < u$ we have $\dim_K[I_{Z+mP}]_{j+1} = \dim_K[I_{Y+mP}]_j = \binom{j+2}{2} - \binom{m+1}{2} - m$ and $\dim_K[I_Z]_{j+1} = \dim_K[I_Y]_j = \binom{j+2}{2} - m$, while for $j + 1 \geq u$ we have $\dim_K[I_{Z+mP}]_{j+1} = \binom{j+3}{2} - \binom{m+1}{2} - m - u = \dim_K[I_Z]_{j+1} - \binom{m+1}{2}$.*

Proof. Let ℓ be a linear form defining L . Then there are exact sequences

$$0 \rightarrow (R/I_Y)(-1) \rightarrow R/I_Z \rightarrow R/(I_Z, \ell) \rightarrow 0$$

and

$$0 \rightarrow (R/I_{Y+mP})(-1) \rightarrow R/I_{Z+mP} \rightarrow R/(I_{Z+mP}, \ell) \rightarrow 0.$$

Since the saturation of both of (I_Z, ℓ) and (I_{Z+mP}, ℓ) is generated by ℓ and a form of degree u , we get $\dim_K[I_{Z+mP}]_{j+1} = \dim_K[I_{Y+mP}]_j$ and $\dim_K[I_Z]_{j+1} = \dim_K[I_Y]_j$ if $j + 1 < u$. Combining Lemmas 2.14 and 2.13, we get $H^1(\mathcal{I}_{Y+mP}(j + 1)) = 0$ if $j \geq m$. Moreover, we have $H^1(\mathcal{I}_Y(j)) = 0$ if $j \geq m$. This gives $H^1(\mathcal{I}_{Z+mP}(j + 1)) = 0 = H^1(\mathcal{I}_Z(j))$ if $j + 1 \geq u$. Now the formulas for the dimensions follow. \square

Corollary 4.13. *Let Z be a reduced 0-dimensional subscheme of \mathbb{P}^2 , and let P be a general point. If Z has an unexpected curve, then the unique unexpected curve of degree $m_Z + 1$ has a unique component of degree more than 1.*

Proof. Lemma 3.6 gives that Z is generally special. Hence, by Proposition 3.7 we know that there is a unique curve C_Z of degree $m_Z + 1$ vanishing at Z and to order m_Z at P and that C_Z has a unique component B_Z of degree $t + 1$ vanishing to order $t \geq 0$ at P . We have to that $t > 0$ if Z has unexpected curves.

Assume, on the contrary, that $t = 0$. Note that then C_Z has m_Z lines through P , and these also go through m_Z points of Z ; let Y denote the scheme of these m_Z points of Z . The only other component of C_Z is B_Z , also a line, and it contains the other $u = |Z| - m_Z$ points of Z . From Proposition 3.7 we have $(m_Z + 1)^2 - m_Z^2 - |Z| \leq -2$, hence $m_Z + 3 \leq |Z| - m_Z = u$. I.e., at least $m_Z + 3$ of the points of Z lie on the line C'_Z . Applying Lemma 4.12 with $m = m_Z$ and $L = C'_Z$ shows that $\dim_K[I_{Z+mP}]_{j+1} = \dim_K[I_{Y+mP}]_j = \dim_K[I_Y]_j - \binom{j+2}{2} = \dim_K[I_Z]_{j+1} - \binom{j+2}{2}$ for $j + 1 < u$ and $\dim_K[I_{Z+mP}]_{j+1} = \dim_K[I_Z]_{j+1} - \binom{m+1}{2}$ when $j + 1 \geq u$. Thus Z has no unexpected curves if $t = 1$. \square

5. LINE ARRANGEMENTS AND MULTIPLICITY INDICES

We now assume that the base field K has characteristic zero. The results that we obtained in the previous sections about sets of points and the unexpected curves associated to them have a beautiful application to the line arrangements dual to those points, advancing the known theory about this connection.

Let \mathcal{A} be a line arrangement in \mathbb{P}^2 ; i.e. \mathcal{A} is a set of lines defined by a reduced product, f , of linear forms. The Jacobian ideal of f , $J = \text{Jac}(f) = (f_x, f_y, f_z)$, fits into an exact sequence of graded R -modules

$$0 \rightarrow F_2 \rightarrow F_1 \rightarrow R(-(\deg f - 1))^3 \rightarrow J \rightarrow 0,$$

where F_1 and F_2 are finitely generated free R -modules. Let D_0 be the cokernel of $F_2 \rightarrow F_1$; the sheafification of D_0 is called the *syzygy bundle*. Sheafifying and twisting by $\deg(f) - 1$ gives an exact sheaf sequence

$$0 \rightarrow \mathcal{D} \rightarrow \mathcal{O}_{\mathbb{P}^2}^3 \rightarrow \mathcal{J}(\deg(f) - 1) \rightarrow 0$$

where \mathcal{D} is the sheafification of $D(f) = D_0(\deg f - 1)$, called the *derivation bundle* of $\mathcal{A} = \mathcal{A}(f)$. Indeed, \mathcal{D} is locally free of rank two. By Grothendieck's theorem, the restriction of \mathcal{D} to a general line has the form $\mathcal{O}_{\mathbb{P}^1}(-a) \oplus \mathcal{O}_{\mathbb{P}^1}(-b)$ for some integers $a \leq b$. We refer to (a, b) as the *splitting type* of \mathcal{A} . Note that $a + b = \deg f - 1$.

Moreover, one says that the arrangement \mathcal{A} is *free* if the module D_0 is a free R -module, which is true if and only if the Jacobian ideal J is saturated. In this case, \mathcal{A} having splitting type (a, b) is equivalent to $D_0(\deg f - 1) \cong R(-a) \oplus R(-b)$, and one also says that \mathcal{A} is *free with exponents* (a, b) . In particular, if J has only two minimal generators, then the arrangement \mathcal{A} is free with exponents $(0, \deg f - 1)$. It is not hard to see that this happens if and only if every line in \mathcal{A} passes through a single point P .

Remark 5.1. A line arrangement \mathcal{A} in \mathbb{P}^2 is *supersolvable* if it has a so-called *modular point*, i.e. a point P with the property that if $\ell_1, \ell_2 \in \mathcal{A}$ and if Q is the intersection of ℓ_1 and ℓ_2 then the line joining P and Q is a line of \mathcal{A} . A standard fact is that if \mathcal{A} is a supersolvable line arrangement consisting of d lines, m of which pass through the modular point P , then \mathcal{A} is free, and the splitting type is $(m - 1, d - m)$. We are grateful to S. Toh  neanu for pointing out that the computation of the splitting type is a simple application of the addition-deletion theorem (Theorem 5.12 below) using induction on d , with the base case being the case that all lines pass through a single point, mentioned above.

The connection to the previous section is given by the following result.

Theorem 5.2 (Faenzi and Vall  s). *Let $\mathcal{A}(f)$ be a line arrangement in $\mathbb{P}^{2\vee}$ with splitting type (a, b) , where $a \leq b$. Let $Z \subset \mathbb{P}^2$ be the set of points dual to the lines in $\mathcal{A}(f)$. Then $m_Z = a$. That is, for a general point $P \in \mathbb{P}^2$,*

$$h^0(\mathcal{I}_Z \otimes \mathcal{I}_P^a)(a + 1) \neq 0 \quad \text{and} \quad h^0(\mathcal{I}_Z \otimes \mathcal{I}_P^{a-1})(a) = 0.$$

Proof. This follows from [FV, Theorem 4.3]. □

Applying Corollary 4.2, there is also an interpretation of b .

Corollary 5.3. *If a line arrangement $\mathcal{A}(f)$ has splitting type (a, b) and Z is the dual set of points, then $b = u_Z + 1$. That is, for a general point $P \in \mathbb{P}^2$,*

$$h^1((\mathcal{I}_Z \otimes \mathcal{I}_P^{b-1})(b)) = 0 \quad \text{and} \quad h^1((\mathcal{I}_Z \otimes \mathcal{I}_P^{b-2})(b - 1)) \neq 0.$$

Proof. Combine Theorem 5.2 and Corollary 4.2. \square

Using the above interpretations of the splitting type, we record some consequences of the results in the previous sections for line arrangements.

Proposition 5.4. *Let \mathcal{A} be an arrangement of lines with splitting type of \mathcal{A} is (a, b) , with $a \leq b$. Let Z be the set of points dual to the lines of \mathcal{A} . Then one has:*

- (a) *Z is generally special if and only if $b - a \geq 2$.*
- (b) *If $b - a \geq 2$, $a \geq 2$, and $[I_{Z+m_Z P}]_{m_Z+1}$ contains an irreducible form then Z admits an unexpected curve.*
- (d) *If $b - a \geq 2$ but there is no irreducible form in $[I_{Z+m_Z P}]_{m_Z+1}$, then Z does not necessarily have an unexpected curve, regardless of the value of $b - a$.*

Proof. Part (a) follows from Theorem 4.1. Part (b) is a consequence of Corollary 4.4, while (c) follows from Proposition 4.3. \square

Remark 5.5. The preceding proposition shows that the condition $b - a \geq 2$ is not enough to guarantee that Z has an unexpected curve. We also note that the existence of an unexpected curve does not depend on the freeness of the syzygy bundle, as demonstrated by Proposition 5.13 and Example 5.11 below, which are concerned with free and non-free arrangements, respectively.

The result mentioned in the introduction follows easily now.

Proof of Theorem 1.4. Combine Theorems 2.16, 5.2, and Corollary 5.3. \square

For another consequence of our results on points we need some preparation. We begin by recalling a very special case of a construction that was introduced in [LR] and that has been generalized quite extensively in the decades that followed. We present only the elementary version that we will use here. See for instance [BM].

Lemma 5.6. *Let I_X be the saturated ideal of a scheme X in \mathbb{P}^2 . Let $f \in I_X$ be a homogeneous polynomial. Let ℓ be a linear form such that ℓ does not vanish on any component of X . Let Y be the complete intersection scheme with homogeneous ideal (ℓ, f) . Then $\ell \cdot I_X + (f)$ is the saturated ideal of the scheme $X \cup Y$.*

Lemma 5.7. *Let $\mathcal{A}(f)$ be a line arrangement of $d \geq 3$ lines, and let ℓ be a linear form defining a line that meets $\mathcal{A}(f)$ in d distinct points. Then the line arrangement $\mathcal{A}(f\ell)$ is free if and only if the d lines of $\mathcal{A}(f)$ all meet in one point.*

Proof. Let $X \subset \mathbb{P}^2$ be the zero-dimensional subscheme defined by the Jacobian ideal of f . By assumption, the Jacobian ideal of $f\ell$ defines the subscheme $X \cup Y$, where Y is the complete intersection defined by the ideal (ℓ, f) . By Lemma 5.6, the homogeneous ideal of $X \cup Y$ is $\ell I_X + (f)$. Since f is not a minimal generator of I_X , the product of ℓ and the minimal generators of I_X together with f form a minimal generating set of $\ell I_X + (f)$ (see [MN, Corollary 4.5]). Hence this ideal has one minimal generator more than I_X .

Now, if $\mathcal{A}(f\ell)$ is free, then $\text{Jac}(f\ell) = \ell I_X + (f)$ has at most three minimal generators. It follows that in this case the ideal I_X has two minimal generators of degree $d - 1$. Since I_X is the saturation of $\text{Jac}(f)$, we get $I_X = \text{Jac}(f)$. As observed above, this means that the lines of $\mathcal{A}(f)$ pass through a single point, as desired.

Conversely, if $\text{Jac}(f)$ is generated by two forms of degree $d - 1$, then the saturation of $\text{Jac}(f\ell)$ has three minimal generators of degree d . It follows that $\text{Jac}(f) = \ell I_X + (f)$ is saturated, that is, $\mathcal{A}(f\ell)$ is free, which completes the argument. \square

Proposition 5.8. *Let \mathcal{A} be an arrangement of d lines in \mathbb{P}^2 . Assume that there is a point Q through which there pass t lines of \mathcal{A} , with $\frac{d+2}{2} \leq t$, and assume that none of the other lines pass through Q . Let Z be the set of points dual to \mathcal{A} . Then:*

- (a) \mathcal{A} has splitting type $(d - t, t - 1)$;
- (b) $m_Z = d - t$;
- (c) Z does not admit an unexpected curve;
- (d) If $\frac{d+3}{2} \leq t$ then Z is generally special;
- (e) If $t \leq d - 2$ and if, at all points of \mathbb{P}^2 other than Q , at most two lines of \mathcal{A} pass, then \mathcal{A} is not free.

Proof. Parts (b) - (d) are a consequence of Proposition 4.3. Theorem 5.2 and (b) give (a).

We now prove (e). Notice first that the assumptions force $d \geq 6$. They also force the number of lines *not* through Q , i.e. $d - t$, to satisfy $2 \leq d - t \leq \frac{d-2}{2}$. Let $\mathcal{B} \subset \mathcal{A}$ be the set of lines through Q . By assumption, \mathcal{A} is obtained from \mathcal{B} by adding successively lines such that each line meets each of the previous lines in distinct points. Since we have to add at least two such lines to \mathcal{B} , Lemma 5.7 shows that \mathcal{A} is not free. \square

We now consider the change of the splitting type if one adds a line to an arrangement.

Proposition 5.9. *Let $\mathcal{A}(f) \subset \mathbb{P}^2$ be an arrangement of d lines with splitting type (a, b) , where $a \leq b$. Let $\ell \in R$ be a linear form. Then one has:*

- (a) *The splitting type of $\mathcal{A}(\ell f)$ is $(a + 1, b)$ or $(a, b + 1)$.*
- (b) *If ℓ is a general linear form, then*
 - $\mathcal{A}(\ell f)$ *has splitting type $(a, a + 1)$ if $(a, b) = (\frac{d-1}{2}, \frac{d-1}{2})$; and*
 - $\mathcal{A}(\ell f)$ *has splitting type $(a + 1, b)$ if $a < \frac{d-1}{2} < b$.*

Proof. This follows by combining Theorem 5.2 and Lemma 4.8 as well as Propositions 4.9 and 4.10. \square

It is worth pointing out that the splitting type of a line arrangement can in principle be computed using computer algebra software.

Remark 5.10. Suppose that we are given the form f defining a line configuration $\mathcal{A}(f)$ and we wish to compute m_Z for the union of points, Z , dual to the lines. This can arise, for example, if we know the coordinates for the individual points; it can also arise if we are given only the ideal of the points, since f can be computed symbolically from the ideal [MP]. Say $\deg f = d$. Let J be the Jacobian ideal (f_x, f_y, f_z) . If $\mathcal{A}(f)$ is free, we have seen above how to compute the splitting type from a minimal free resolution of J .

Now assume that $\mathcal{A}(f)$ is not free. Let ℓ be a general linear form. Compute a minimal free resolution of $\bar{J} = \frac{J + (\ell)}{(\ell)}$ over \bar{R} . It has the form

$$0 \rightarrow \mathbb{F}_2 \rightarrow \bar{R}^3(-d + 1) \rightarrow \bar{J} \rightarrow 0,$$

where \mathbb{F}_2 (necessarily) has rank 2. If it is $\bar{R}(-a) \oplus \bar{R}(-b)$, then the (generic) splitting type is $(a + 1 - d, b + 1 - d)$.

This computation depends on restricting modulo a *general* linear form, since the Betti numbers of the restriction can depend on the linear form. We address two computational approaches.

First, let ℓ be a linear form chosen with “random” coefficients. With very high probability ℓ will serve as a general linear form for the purpose of computing the splitting type. An

alternative approach is computationally much more expensive, but has the advantage of being mathematically certain. That is to resolve \bar{J} over the ring $\mathbb{Q}(a, b)[x, y]$, where $\ell = ax + by + z$. For relatively small configurations, this approach is effective.

To shorten the computation, in many cases it may be possible to combine ad hoc arguments and computer calculations, as in the following example.

Example 5.11. Consider the non-free configuration of 19 lines given in Example 3.10. There we saw that the splitting type of this arrangement is $(8, 10)$ or $(9, 9)$. We are going to show that it is the former.

For a general linear form ℓ , set $\bar{R} = R/\ell R$ and $\bar{J} = \frac{J+(\ell)}{(\ell)}$, where $J \subset R$ is the Jacobian ideal. Consider the graded exact sequence induced by multiplication by ℓ

$$(R/J)(-1) \xrightarrow{\ell} R/J \rightarrow \bar{R}/\bar{J} \rightarrow 0.$$

Using a computer algebra system, one gets $\dim_K[R/J]_{25} = 243$ and $\dim_K[R/J]_{26} = 244$. Hence, the above exact sequence, considered in degree 26, gives $[\bar{R}/\bar{J}]_{26} \neq 0$. The minimal free resolution of \bar{R}/\bar{J} over \bar{R} has the form

$$0 \rightarrow \mathbb{F}_2 \rightarrow \bar{R}^3(-18) \rightarrow \bar{R} \rightarrow \bar{R}/\bar{J} \rightarrow 0.$$

Since $[\bar{R}/\bar{J}]_{26} \neq 0$, we obtain $[\mathbb{F}_2]_{26} \neq 0$. It follows that the splitting type is $(26-18, 28-18) = (8, 10)$ as claimed.

It is interesting to note that replacing $2x + y$ by $2y - x$ in the configuration of 19 lines yields a free configuration with splitting type $(7, 11)$.

There are some further theoretical tools for determining splitting types, which we consider now. Let $\mathcal{A} = \mathcal{A}(f)$ be a line arrangement in \mathbb{P}^2 . Let L be one of the components of \mathcal{A} defined by a linear form ℓ . Let $g = f/\ell$. Then \bar{g} , the restriction of g to L , is a polynomial of the same degree as g though it is not necessarily reduced. If \bar{g}' is the radical of \bar{g} , then \bar{g}' defines a hyperplane arrangement of $L = \mathbb{P}^1$, called the *restriction*, which we denote \mathcal{A}'' . Moreover, the arrangement defined by g is often denoted by \mathcal{A}' , and one refers to $(\mathcal{A}', \mathcal{A}, \mathcal{A}'')$ as a triple of hyperplane arrangements. Thus if \mathcal{A} is a line arrangement then \mathcal{A}' is obtained from \mathcal{A} by removing a line L , and \mathcal{A}'' is the restriction of \mathcal{A}' to L . Notice that the arrangement $\mathcal{A}'' \subset \mathbb{P}^1$ is free with exponent $|\mathcal{A}''| - 1$.

Theorem 5.12 (Addition-Deletion Theorem; see, e.g., [OT, Theorem 4.51]). *Let $(\mathcal{A}', \mathcal{A}, \mathcal{A}'')$ be a triple of line arrangements. Then any two of the following imply the third:*

- \mathcal{A} is free with exponents $(a+1, b)$ or $(a, b+1)$;
- \mathcal{A}' is free with exponents (a, b) ;
- \mathcal{A}'' is free with exponent (b) or (a) (i.e. \mathcal{A}' meets ℓ in $b+1$ or $a+1$ points, ignoring multiplicity).

We use this result to study so-called *Fermat arrangements* of lines [U]. We note that these are also sometimes known as *monomial arrangements* (see [Su, Example 10.6] and [OT, page 247]). These arrangements consist of $3t$ lines ($t \geq 1$) that are defined by the linear factors of $f = (x^t - y^t)(x^t - z^t)(y^t - z^t)$. If $t > 3$ or $t = 2$, there are t^2 points where exactly 3 lines cross and 3 points where exactly t lines cross, and no other crossing points. When $t = 3$, there are 12 points where exactly 3 lines cross and no other crossing points. When $t = 1$ there is only one crossing point, and 3 lines cross there. The set of points Z dual to the lines is defined by the ideal $(x^t + y^t + z^t, xyz)$ (i.e., the intersection of the Fermat t -ic with the coordinate axes)

when t is odd, and by $(x^t - y^t, z) \cap (x^t - z^t, y) \cap (y^t - z^t, x)$ when t is even. Although the freeness is known (and the splitting types too, in terms of degrees of generators of certain rings of invariants) [OT, Theorem 6.60, & p. 247], for the reader's convenience, we include a short proof here as part of the next result.

Proposition 5.13. *If $t > 2$, then the Fermat line configuration is free, with splitting type $(t + 1, 2t - 2)$. If $t \geq 5$, the dual set of points admits unexpected curves of degrees $t + 2, \dots, 2t - 3$.*

Proof. We first prove freeness. We will start with a slightly larger line arrangement, and produce the Fermat arrangement by removing two lines. The configuration of lines defined by the factors of $g = xy(x^t - y^t)(x^t - z^t)(y^t - z^t)$ is supersolvable since every point of intersection of two of the lines is on one of the lines through the point defined by $x = 0$ and $y = 0$. Thus the line arrangement $\mathcal{A}(g)$ is free (see Remark 5.1).

Now we determine its splitting type, (a, b) , where $a \leq b$. Observe that there are $d = 3t + 2$ lines in $\mathcal{A} = \mathcal{A}(g)$, and the modular point lies on $m = t + 2$ lines. Hence by Remark 5.1, the splitting type of \mathcal{A} is $(t + 1, 2t)$.

Next we successively remove the lines defined by x and y from \mathcal{A} . First let $\mathcal{A}' = \mathcal{A}(\frac{g}{x})$ and let \mathcal{A}'' be the arrangement obtained by restricting \mathcal{A}' to $x = 0$. Clearly \mathcal{A}'' is free with type $t + 1$, so by the Addition-Deletion Theorem 5.12 \mathcal{A}' is an arrangement which is free of type $(t + 1, 2t - 1)$. Now delete y from \mathcal{A}' and apply Addition-Deletion again to see that $(x^t - y^t)(x^t - z^t)(y^t - z^t)$ gives a free arrangement of type $(t + 1, 2t - 2)$.

Using Theorem 5.2 and Corollary 5.3, we conclude for the dual set of points Z that $m_Z = t + 1$ and $u_Z = 2t - 3$. By [Ha, Theorem III.1(a)], the $3t$ points of Z impose independent conditions on forms of degree $t + 1$ or more, so $h^0(\mathcal{I}_Z(j + 1)) = \binom{j+3}{2} - 3t$ for $j + 1 \geq t + 1$. Thus, taking $j = m_Z = t + 1$, we have $h^0(\mathcal{I}_Z(t + 2)) - \binom{t+2}{2} = 5 - t$ and since $t_Z \geq m_Z$, we see $t_Z > m_Z$ for $t \geq 5$. Now Theorem 2.16 gives that, for $t \geq 5$, the set Z admits an unexpected curve of degree j whenever $t + 2 \leq j \leq 2t - 2$. \square

In order to derive our next results we need the concept of a stable vector bundle. For unexplained terminology on vector bundles we refer to [OSS]. Stable vector bundles of rank two can be characterized cohomologically.

Lemma 5.14 ([H, Lemma 3.1]). *A reflexive sheaf \mathcal{F} of rank two over \mathbb{P}^n is stable if and only if $H^0(\mathcal{F}_{\text{norm}}) = 0$. If $c_1(\mathcal{F})$ is even, then \mathcal{F} is semistable iff $H^0(\mathcal{F}_{\text{norm}}(-1)) = 0$. If $c_1(\mathcal{F})$ is odd then semistability and stability coincide.*

Stability is related to the existence of unexpected curves as we see now.

Proposition 5.15. *Let \mathcal{A} be a line arrangement with splitting type (a, b) , where $a \leq b$. If the dual set of points Z admits an unexpected curve, then $b \geq a + 2$. In particular, the derivation bundle of \mathcal{A} is not semistable.*

Proof. If the derivation bundle of \mathcal{A} is semistable, then the Grauert-Mulich theorem [GM] gives $b - a \leq 1$. However, if Z has an unexpected curve, then it is generally special by Lemma 3.6, and so Proposition 5.4(a) gives $b - a \geq 2$. \square

The following result is useful for establishing stability.

Lemma 5.16. *Let \mathcal{A} be $(\mathcal{A}', \mathcal{A}, \mathcal{A}'')$ a triple of line arrangements, where \mathcal{A} consists of d lines. Then one has:*

- (a) ([S, Theorem 4.5(a)]) If d is odd, then \mathcal{D} is stable if \mathcal{D}' is stable and $|\mathcal{A}''| > \frac{d+1}{2}$.
- (b) If d is odd, then \mathcal{D} is semistable if \mathcal{D}' is stable.
- (c) ([S, Theorem 4.5(c)]) If d is even, then \mathcal{D} is stable if \mathcal{D}' is semistable and $|\mathcal{A}''| > \frac{d}{2}$.
- (d) If d is even, then \mathcal{D} is stable if \mathcal{D}' is stable.

Proof. According to [S, Theorem 3.2], there is an exact sequence

$$0 \rightarrow \mathcal{D}'(-1) \rightarrow \mathcal{D} \rightarrow \mathcal{O}_{\mathbb{P}^1}(1 - |\mathcal{A}''|) \rightarrow 0.$$

It implies parts (a) and (c). Using that for any vector bundle \mathcal{E} of rank two on \mathbb{P}^2 one has $\mathcal{E}^\vee \cong \mathcal{E}(c_1(\mathcal{E}))$, dualizing gives the exact sequence (see also [FV, Proposition 5.1])

$$(5.1) \quad 0 \rightarrow \mathcal{D} \rightarrow \mathcal{D}' \rightarrow \mathcal{O}_{\mathbb{P}^1}(-d + |\mathcal{A}''| + 1) \rightarrow 0.$$

Applying Lemma 5.14, parts (b) and (d) follow. \square

Remark 5.17. Lemma 5.16(b) improves [S, Theorem 4.5(b)] by eliminating any assumption on \mathcal{A}'' . Note that in this case stability and semistability of \mathcal{D}' are equivalent by Lemma 5.14.

As a first consequence, we get information on sufficiently general line arrangements.

Proposition 5.18. *Let \mathcal{A}_d be a configuration of d lines in \mathbb{P}^2 such that no three lines of \mathcal{A}_d meet in a point. Then the splitting type for \mathcal{A}_d is*

$$\left(\left\lfloor \frac{d-1}{2} \right\rfloor, \left\lceil \frac{d-1}{2} \right\rceil \right).$$

Moreover, \mathcal{A}_d is free if and only if $d \leq 3$.

Proof. Let J_d be the Jacobian ideal of \mathcal{A}_d and let \bar{J}_d be its saturation. By assumption, the lines in \mathcal{A}_d form a star configuration. Thus, by [GHM] we know that the minimal free resolution of \bar{J}_d is

$$0 \rightarrow R(-d)^{n-1} \rightarrow R(-d+1)^n \rightarrow \bar{J}_d \rightarrow 0.$$

In particular, J_d is saturated if and only if $d \leq 3$, so \mathcal{A}_d is free if and only if $d \leq 3$.

Let us establish some notation. This minimal free resolution for J_d truncates to a short exact sequence

$$0 \rightarrow E_d \rightarrow R(-d+1)^3 \rightarrow J_d \rightarrow 0.$$

Let \mathcal{E}_d be the sheafification of the reflexive module E_d . Then $\mathcal{D}_d = \mathcal{E}_d(d-1)$ is the derivation bundle of \mathcal{A}_d . Note also that $(\mathcal{D}_d)_{\text{norm}} = \mathcal{E}_d(\frac{3d-3}{2})$ when d is odd, and $(\mathcal{D}_d)_{\text{norm}} = \mathcal{E}_d(\frac{3d-4}{2})$ if d is even.

First consider $d = 3$. Then \mathcal{A}_d is free and we have the minimal free resolution

$$0 \rightarrow R(-3)^2 \rightarrow R(-2)^3 \rightarrow J_3 \rightarrow 0.$$

Thus $\mathcal{E}_3 = \mathcal{O}_{\mathbb{P}^2}(-3)^2$, $\mathcal{D}_3 = \mathcal{O}_{\mathbb{P}^2}(-1)^2$ and $(\mathcal{D}_3)_{\text{norm}} = \mathcal{O}_{\mathbb{P}^2}^2$. By Lemma 5.14, \mathcal{D}_3 is semistable. Clearly the exponent set for \mathcal{A}_3 is $(1, 1)$ as claimed.

Now assume that $d = 4$. It follows from Lemma 5.16 that \mathcal{D}_4 is stable, so the splitting type is as claimed thanks to the Grauert-Mulich theorem [GM].

Using Lemma 5.16, we obtain by induction that \mathcal{D}_d is stable for all $d \geq 4$. Hence by the Grauert-Mulich theorem, the splitting type of \mathcal{D}_d is as claimed. \square

This has the following consequence for the dual set of points. Recall that a set of points in \mathbb{P}^2 is said to be in *linearly general position* if no three of its points are on a line.

Corollary 5.19. *Let Z be a set of points in \mathbb{P}^2 in linear general position. Then $m_Z = \left\lfloor \frac{|Z|-1}{2} \right\rfloor$, $u_Z = \left\lceil \frac{|Z|-1}{2} \right\rceil - 1$, and Z does not admit an unexpected curve. Furthermore, for a general point P ,*

$$h^0(\mathcal{I}_Z \otimes \mathcal{I}_P^{m_Z})(m_Z + 1) = \begin{cases} 2 & \text{if } Z \text{ is odd;} \\ 1 & \text{if } Z \text{ is even,} \end{cases}$$

and $[I_{Z+m_Z P}]_{m_Z+1}$ contains an irreducible form.

Proof. Notice that a set of points is in linearly general position if and only if the set of dual lines has the property that no three of them meet in a point. Hence, Proposition 5.18 gives the asserted values of m_Z and u_Z . Combined with Theorem 4.1, we get the claim about $h^0(\mathcal{I}_Z \otimes \mathcal{I}_P^{m_Z})(m_Z + 1)$. Furthermore, Corollary 2.10 implies $m_Z = t_Z$, and hence Z does not admit an unexpected curve by Theorem 4.5. It remains to show the irreducibility statement.

First, assume Z is even. Then we have seen that, for each point $Q \in Z$, one has $m_Z = \frac{|Z|-2}{2} = m_{Z-Q}$. Hence, the unique curve determined by $[I_{Z+m_Z P}]_{m_Z+1}$ is irreducible by Corollary 4.11.

Second, assume Z is odd. Choose a point $Q \notin Z$ such that $Z + Q$ is in linearly general position. Then $m_Z = \frac{|Z|-1}{2} = m_{Z+Q}$, and so $[I_{Z+Q+m_Z+Q P}]_{m_Z+Q+1} \subset [I_{Z+m_Z P}]_{m_Z+1}$. We just showed that the space on the left-hand side contains an irreducible form, which completes the argument. \square

Remark 5.20. Corollary 5.19 is a statement about a set of points. It would be interesting to have a more direct proof and to decide if the conclusion is also true if the base field has positive characteristic.

In [DIV, Proposition 7.3] the authors consider the B_3 line configuration and remark that the dual set of points lies on a curve of degree 4 vanishing to order 3 at a general point. They state without proof that this curve is irreducible. We remark that this is almost immediate (see the proof of Proposition 5.22 below).

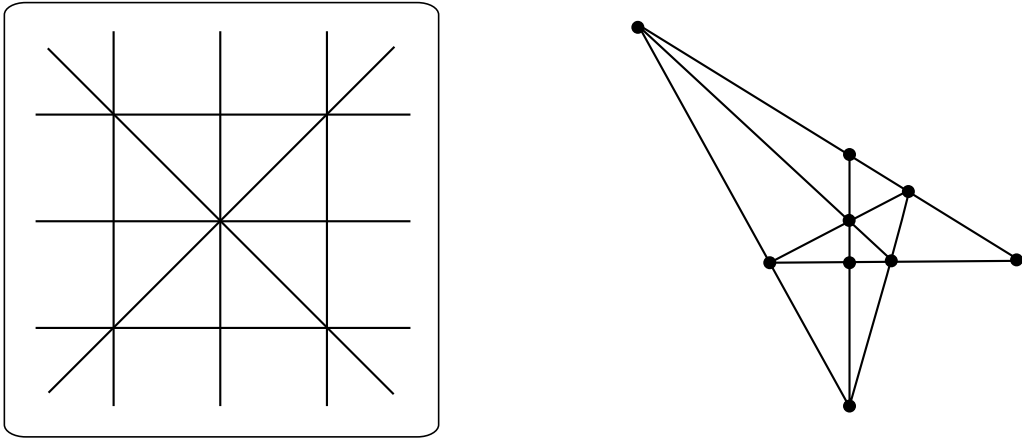


FIGURE 3. The B_3 configuration and its dual set of points.

We generalize this example now. First we will describe the family of line arrangements that we will study, and then we will discuss the associated configurations of points.

Example 5.21. Let \mathcal{A} be the arrangement of five lines defined by the form $xyz(x+y)(x-y)$. We will denote by a the line $x - y = 0$, by d the line $x + y = 0$, by i the line at infinity ($z = 0$), and by h_1 and v_1 the x and y axes, respectively. We remark in passing that there is some flexibility in the choice of these five lines, but that an arbitrary configuration of five lines with the same intersection lattice is *not* always going to lead to arrangements with the properties that we will describe. (For example, replacing $x - y$ by any other line through the origin will fail to satisfy the requirement below that h_3 passes through $d \cap v_2$.)

We will add lines to \mathcal{A} , and define the line arrangements \mathcal{A}_k inductively, where k is the total number of lines that we have added to \mathcal{A} . In what follows, for simplicity we will refer to the lines containing the point of intersection of i and v_1 as “vertical lines,” and the lines containing the point of intersection of i and h_1 as “horizontal lines.”

\mathcal{A}_1 is obtained by adding to \mathcal{A} an arbitrary vertical line, v_2 . The next three lines added to \mathcal{A}_1 are then determined: h_2 is the horizontal line through $a \cap v_2$, v_3 is the vertical line through $d \cap h_2$, and h_3 is the horizontal line through $a \cap v_3$. The key fact is that h_3 also passes through $d \cap v_2$. This gives the arrangements $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4$.

We continue in this way, taking an arbitrary vertical line v_4 and adding a horizontal line h_4 , a vertical line v_5 , and another horizontal line h_5 in the manner just described to obtain configurations $\mathcal{A}_5, \mathcal{A}_6, \mathcal{A}_7, \mathcal{A}_8$. Of special interest to us will be the configurations \mathcal{A}_n where n is a multiple of 4. In particular, $\mathcal{A}_{4(k+1)}$ is obtained from \mathcal{A}_{4k} by adding the lines $v_{2k+2}, h_{2k+2}, v_{2k+3}, h_{2k+3}$ in that order. See Figure 4 for an example of the line configuration, and Figure 5 for an example of the set of points dual to such a configuration, where we have used the same names for the points as we used for the lines that they are dual to.

Notice that \mathcal{A}_4 is the B_3 arrangement.

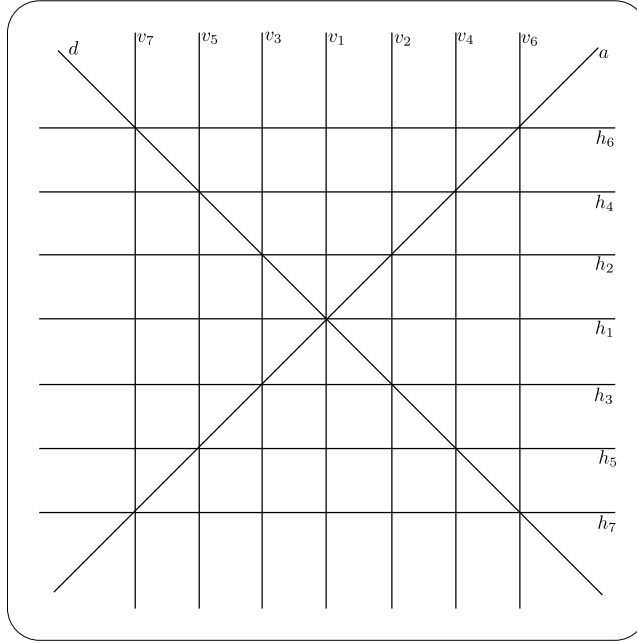
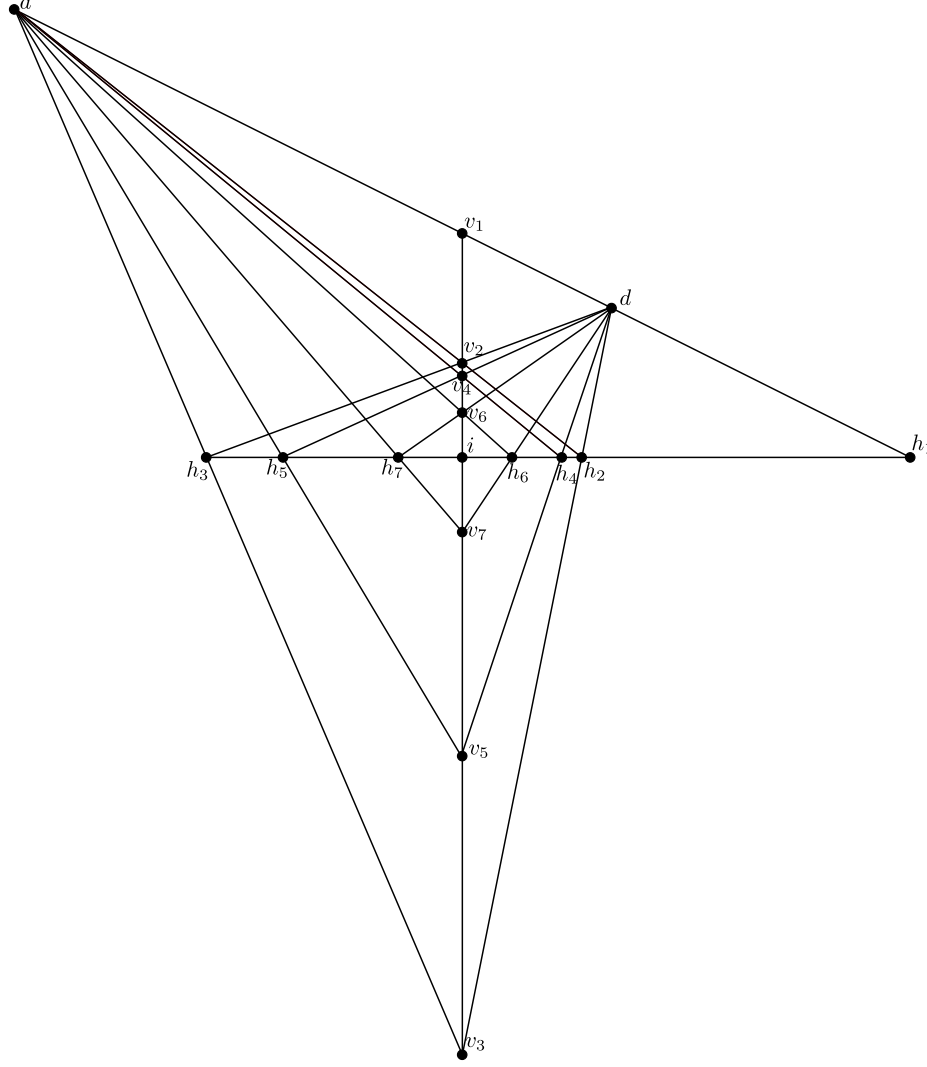


FIGURE 4. The line arrangement \mathcal{A}_{12} .

One can easily check using Theorem 5.12 that these configurations are all free, with splitting types as follows:

- $(2k + 1, 2k + 3)$ for \mathcal{A}_{4k} ;


 FIGURE 5. The point configuration dual to \mathcal{A}_{12} .

- $(2k + 2, 2k + 3)$ for \mathcal{A}_{4k+1} ;
- $(2k + 3, 2k + 3)$ for \mathcal{A}_{4k+2} ;
- $(2k + 3, 2k + 4)$ for \mathcal{A}_{4k+3} ;

Let us denote by Z_n the set of $n + 5$ points dual to the line arrangement \mathcal{A}_n .

Proposition 5.22. *If $k \geq 1$, then Z_{4k} has multiplicity index $m_{Z_{4k}} = 2k + 1$, speciality index $u_{Z_{4k}} = 2k + 2$, and Z_{4k} admits a unique unexpected curve. It is irreducible and has degree $m_{Z_{4k}} + 1 = 2k + 2$.*

Proof. Since \mathcal{A}_{4k} has splitting type $(2k + 1, 2k + 3)$, we get the claimed values of $m_{Z_{4k}}$ and $u_{Z_{4k}}$. Now Theorem 2.16 gives that Z_{4k} admits an unexpected curve and that it must have degree $2k + 2$. Uniqueness of the unexpected curve is a consequence of Theorem 4.1. It remains to show its irreducibility.

To this end we use Corollary 4.11. It shows that we are done once we have proven that removing any line L from the arrangement \mathcal{A}_{4k} gives an arrangement $\mathcal{A}_{4k} \setminus L$, with splitting type $(2k + 1, 2k + 2)$.

First, let L be any line of \mathcal{A}_{4k} other than the line at infinity i , defined by $z = 0$. Then L meets the other lines of \mathcal{A}_{4k} in $2k + 2$ points. Hence Addition-Deletion yields that $\mathcal{A}_{4k} \setminus L$ is a free arrangement with splitting type $(2k + 1, 2k + 2)$, as claimed.

Second, consider the line i , and set $\mathcal{A}' = \mathcal{A}_{4k} \setminus i$. The line i meets the lines in \mathcal{A}' in four points. Hence, if $k = 1$ (i.e., \mathcal{A}_4 is the B3 configuration), then we conclude as in the first case that \mathcal{A}' has splitting type $(3, 4)$, as desired. Let $k \geq 2$. Now we need a different argument.

Let h be the product of $4k + 3$ linear forms such that $\mathcal{A}_{4k} = \mathcal{A}(z(x^2 - y^2)h)$, and so $\mathcal{A}' = \mathcal{A}((x^2 - y^2)h)$. As observed above, the arrangement $\mathcal{A}(z(x - y)h)$ is free with splitting type $(2k + 1, 2k + 2)$. Since the line defined by $x - y$ meets $\mathcal{A}(zh)$ in $2k + 2$ points, we see that $\mathcal{A}(zh)$ is free with splitting type $(2k + 1, 2k + 1)$. The line $z = 0$ meets the lines of $\mathcal{A}(h)$ in two points. Hence, the logarithmic bundles are related by the exact sequence (see Sequence 5.1)

$$0 \rightarrow \mathcal{D}(hz) \rightarrow \mathcal{D}(h) \rightarrow \mathcal{O}_{\mathbb{P}^1}(-4k) \rightarrow 0.$$

Since $\mathcal{D}(hz) \cong \mathcal{O}_{\mathbb{P}^2}^2(-2k - 1)$ and $\mathcal{D}(h)_{\text{norm}} = \mathcal{D}(h)(2k)$, we conclude that $H^0(\mathcal{D}(h)_{\text{norm}}) = 0$, and so $\mathcal{D}(h)$ is stable by Lemma 5.14. Now Lemma 5.16 shows that $\mathcal{A}((x - y)h)$ is semistable. Hence, its splitting type is $(2k + 1, 2k + 1)$ by the Grauert-Mulich theorem. We have already seen that $\mathcal{A}_{4k} = \mathcal{A}(z(x^2 - y^2)h)$ has splitting type $(2k + 1, 2k + 3)$. Using this information, Proposition 5.9(a) yields that $\mathcal{A}' = \mathcal{A}((x^2 - y^2)h)$ has splitting type $(2k + 1, 2k + 2)$. This completes the argument. \square

Remark 5.23. Let $Z \subset \mathbb{P}^2$ be a set of points with $2m_Z + 2 \geq |Z|$. Let P be a general point. We know from Theorem 4.1 that $[I_{Z+m_Z P}]_{m_Z+1}$ determines a unique curve. This curve depends on P , and only the degree is necessarily invariant as P moves. Let us call the curve C_P . Lemma 4.8 shows that if $Q \in C_P$ then $m_{Z+Q,P} = m_Z$. Notice that this is not necessarily equal to m_{Z+Q} . However, if there is a point $Q \notin Z$ such that $Q \in \bigcap_{P \in \mathbb{P}^2} C_P$ then we do obtain $m_Z = m_{Z+Q}$.

We find it very surprising that such a point Q can exist, i.e. that there can be a new point common to every curve in the family $\{C_P\}$ (which is not a linear system) as P varies in \mathbb{P}^2 . Nevertheless, Corollary 4.11 shows that this has to happen even for *each* point Q of Z when passing from $Z - Q$ to Z , provided $2m_Z + 3 \geq |Z|$ and the curve C_P is irreducible. Indeed, the converse is true as well, and we used it to prove the irreducibility of the unexpected curve in Proposition 5.22.

6. THE STRONG LEFSCHETZ PROPERTY

We now relate the existence of an unexpected curve to the failure of a Lefschetz property. Lefschetz properties of graded algebras were formalized and studied in [HMNW], although as the name suggests, the idea behind it has been a fundamental concept for at least a century. Deciding the presence of Lefschetz properties often is a subtle and challenging problem.

Definition 6.1 ([HMNW]). Let A be a graded artinian quotient of $R = k[x_1, \dots, x_n]$. We say that A has the *Weak Lefschetz Property* (WLP) if multiplication by a general linear form induces a homomorphism of maximal rank from any component of A to the next. We say that A has the *Strong Lefschetz Property* (SLP) if the analogous homomorphism induced by L^k has maximal rank, for all $k \geq 1$. Specializing this, we say that A has the SLP *at range k in degree m* if the homomorphism $\times L^k : A_m \rightarrow A_{m+k}$ has maximal rank.

We recall the following important result.

Theorem 6.2 ([EI]). *Let \wp_1, \dots, \wp_m be ideals of m distinct points in \mathbb{P}^{n-1} . Choose positive integers a_1, \dots, a_m , and let $(l_1^{a_1}, \dots, l_m^{a_m}) \subset R = k[x_1, \dots, x_n]$ be the ideal generated by powers of the linear forms that are dual to the points \wp_i . Then for any integer $j \geq \max\{a_i\}$,*

$$\dim_K [R/(l_1^{a_1}, \dots, l_m^{a_m})]_j = \dim_K [\wp_1^{j-a_1+1} \cap \dots \cap \wp_m^{j-a_m+1}]_j.$$

Remark 6.3. As mentioned earlier, the main inspiration for this paper was the beautiful paper [DIV]. However, at the end of the paper there are two results which are not quite true as stated, and with the results of our paper we can see that the reason is precisely the difference between a set of points admitting an *unexpected* curve and merely being generally special. This is summarized in Corollary 6.6 below, with our corrected statement (and generalization) given in Theorem 6.5. The two results from [DIV] are the following.

[DIV, Proposition 7.2]: *Let $I \subset R = \mathbb{C}[x, y, z]$ be an artinian ideal generated by $2d + 1$ polynomials l_1^d, \dots, l_{2d+1}^d , where l_i are distinct linear forms in \mathbb{P}^2 . Let $Z = \{l_1^\vee, \dots, l_{2d+1}^\vee\}$ be the corresponding set of points in $\mathbb{P}^{2\vee}$. Then the following conditions are equivalent:*

- (i) *The ideal I fails the SLP at the range 2 in degree $d - 2$.*
- (ii) *The derivation bundle $D_0(Z)$ is unstable.*

[DIV, Proposition 7.4]: *Let $\mathcal{A} = \{l_1, \dots, l_{a+b+1}\}$ be a line arrangement that is free with exponents (a, b) such that $a \leq b$, $b - a \geq 2$ and $a + b$ even. The ideal $I = (l_1^{(a+b)/2}, \dots, l_{a+b+1}^{(a+b)/2})$ fails the SLP at the range 2 and degree $(a + b)/2 - 1$.*

We first give a counterexample to the two quoted statements from [DIV].

Example 6.4. Let $1 \leq a \leq b - 1$. Define the arrangement $\mathcal{A}_{a,b}$ by the lines

$$\begin{aligned} & z, \\ & x, x + z, x + 2z, \dots, x + (a - 1)z, \\ & y, y + z, y + 2z, \dots, y + (b - 1)z \end{aligned}$$

It is easy to see $\mathcal{A}_{a,b}$ is supersolvable hence free. Moreover, using addition-deletion (or Remark 5.1) it is easy to see that the splitting type is (a, b) .

Let Z be the set of points dual to these lines. For a concrete example, we will take $a = 3$ and $b = 13$ (see Figure 6).

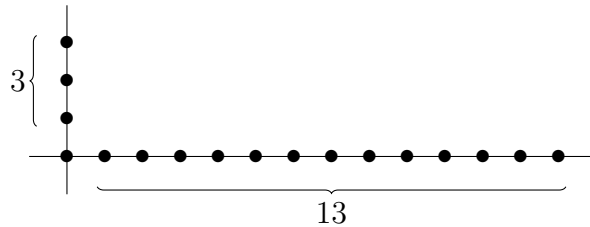


FIGURE 6. The points Z dual to $\mathcal{A}_{3,13}$.

Clearly, [DIV, Propositions 7.2 and 7.4] apply to the configuration $\mathcal{A}_{a,b}$ with $a = 3, b = 13, d = 8$. The splitting type in question is $(3, 13)$; thus the derivation bundle is unstable.

So we consider the ideal

$$I = \langle x^8, (x + z)^8, (x + 2z)^8, y^8, (y + z)^8, \dots, (y + 12z)^8, z^8 \rangle.$$

Its Hilbert function is

$$[1, 3, 6, 10, 15, 21, 28, 36, 33, 27, 19, 12, 7, 3, 1],$$

as can be verified either on a computer or by hand. For a general linear form L , the Hilbert function of $R/(I, L^2)$ is

$$[1, 3, 5, 7, 9, 11, 13, 15, 5].$$

Since

$$[R/I]_{i-2} \xrightarrow{\times L^2} [R/I]_i \rightarrow [R/(I, L^2)]_i \rightarrow 0$$

is exact, a comparison of these two Hilbert functions shows that $\times L^2 : [R/I]_{i-2} \rightarrow [R/I]_i$ has maximal rank for all i . Thus R/I *does* have SLP in range 2.

The error seems to be the subtle point that we have studied in this paper, namely the difference between the existence of a curve containing a set of points, with a certain multiplicity at a general point, and the existence of an *unexpected* curve with these properties. Applying their proof to this example, they note that $h^0(\mathcal{I}_Z \otimes I_P^7)(8) \neq 0$, which is true (in fact, the value is 5). But

$$5 = h^0((\mathcal{I}_Z \otimes I_P^7)(8)) = \dim_K[R/(L_1^8, \dots, L_{17}^8, L^2)]_8$$

and 5 is exactly the difference in the dimensions of the components of R/I in degrees 6 and 8, so the failure of surjectivity by 5 is *expected* and not an indication that the SLP fails there.

Theorem 6.5. *Let $\mathcal{A}(f)$ be a line arrangement in \mathbb{P}^2 , where $f = L_1 \cdots L_d$, and let Z be the set of points in \mathbb{P}^2 dual to these lines. Then Z has an unexpected curve of degree j if and only if $R/(L_1^{j+1}, \dots, L_d^{j+1})$ fails the SLP in range 2 and degree $j - 1$.*

Proof. Let P be a general point in \mathbb{P}^2 , and let L be the linear form dual to P . Consider the multiplication map

$$\times L^2 : [R/(L_1^{j+1}, \dots, L_d^{j+1})]_{j-1} \rightarrow [R/(L_1^{j+1}, \dots, L_d^{j+1})]_{j+1}.$$

Clearly $\dim_K[R/(L_1^{j+1}, \dots, L_d^{j+1})]_{j-1} = \binom{j+1}{2}$. By Macaulay duality,

$$\dim_K[R/(L_1^{j+1}, \dots, L_d^{j+1})]_{j+1} = h^0(\mathcal{I}_Z(j+1)).$$

Hence, the expected dimension of the cokernel is $\max\{h^0(\mathcal{I}_Z(j+1)) - \binom{j+1}{2}, 0\}$. In other words, $R/(L_1^{j+1}, \dots, L_d^{j+1})$ fails the SLP in range 2 and degree $j - 1$ if and only if

$$\dim_K(\text{coker}(\times L^2)) > \max\left\{h^0(\mathcal{I}_Z(j+1)) - \binom{j+1}{2}, 0\right\}.$$

Now, the cokernel of the considered multiplication by L^2 is $[R/(L_1^{j+1}, \dots, L_d^{j+1}, L^2)]_{j+1}$. By Theorem 6.2, its dimension is $h^0((\mathcal{I}_Z \otimes I_P^j)(j+1))$. Thus, we have shown that $R/(L_1^{j+1}, \dots, L_d^{j+1})$ fails the SLP in range 2 and degree $j - 1$ if and only if

$$h^0((\mathcal{I}_Z \otimes I_P^j)(j+1)) > \max\left\{h^0(\mathcal{I}_Z(j+1)) - \binom{j+1}{2}, 0\right\},$$

that is, Z admits an unexpected curve of degree $j + 1$. □

Example 6.4 makes it clear that the failure of the syzygy bundle to be stable is not enough to conclude that the ideal $I = (L_1^d, \dots, L_{|Z|}^d)$ fails the SLP at range 2 in any degree. Nevertheless, there is one additional hypothesis that does allow this conclusion.

Corollary 6.6. *Let $\mathcal{A}(f)$ be a line arrangement in \mathbb{P}^2 with splitting type (a, b) , where $2 \leq a \leq b$. Let $f = L_1 \cdots L_d$, and let Z be the set of points in \mathbb{P}^2 dual to these lines. Assume that $b - a \geq 2$. If in addition there is an irreducible curve of degree $a + 1$ containing $Z + aP$ then $R/(L_1^{a+1}, \dots, L_d^{a+1})$ fails the SLP at range 2 in degree $a - 1$.*

Proof. This follows from Theorem 6.5, via Proposition 5.4, recalling that $a = m_Z$. \square

7. TERAQ'S CONJECTURE

It is natural to wonder to what extent numerical invariants of a line arrangement are determined by its combinatorial properties. The latter are captured by the *incidence lattice* of the arrangement. It consists of all intersections of lines, ordered by reverse inclusion. For example, if $\mathcal{A}(f)$ and $\mathcal{A}(g)$ are two line arrangements in \mathbb{P}^2 with the same incidence lattice, then it follows that the Jacobian ideals of f and g have the same degree.

One of the main open problems is to decide whether freeness of hyperplane arrangements is a combinatorial property. It is open even for line arrangements.

Conjecture 7.1 (Terao). *Freeness of a line arrangement depends only on its incidence lattice.*

Here we want to use our earlier results to state an equivalent version of this conjecture. We need some preparation.

Consider a vector bundle \mathcal{E} on \mathbb{P}^2 of rank two. As pointed out above, its restriction to a general line L has the form $\mathcal{O}_L(-a) \oplus \mathcal{O}_L(-b)$ for some integers $a \leq b$. The pair (a, b) is the (generic) *splitting type* of \mathcal{E} . If \mathcal{E} splits as a direct sum of line bundles, then $c_2(\mathcal{E}) = ab$, where $c_2(\mathcal{E})$ denotes the second Chern class of \mathcal{E} . The converse is true as well.

Theorem 7.2 ([Y, Theorem 1.45]). *For every rank two vector bundle \mathcal{E} on \mathbb{P}^2 with generic splitting type (a, b) , one has $c_2(\mathcal{E}) \geq ab$. Furthermore, equality is true if and only if \mathcal{E} splits as a direct sum of line bundles*

Recall that the derivation bundle $\mathcal{D}(f)$ of a line arrangement $\mathcal{A}(f)$ is the sheafification of the module $D(f)$, defined by the exact sequence

$$0 \rightarrow D(f) \rightarrow R^3 \rightarrow \text{Jac}(f)(\deg f - 1) \rightarrow 0.$$

It follows that

$$(7.1) \quad c_2(\mathcal{D}(f)) = (\deg f - 1)^2 - \deg \text{Jac}(f).$$

We are ready to establish the following result, which is implicitly used in [DIV].

Proposition 7.3. *Let $\mathcal{A}(f)$ and $\mathcal{A}(g)$ be two line arrangements with the same incidence lattice. Assume $\mathcal{A}(f)$ is free with splitting type (a, b) . Then one has:*

- (a) $\mathcal{A}(g)$ is free if and only if $\mathcal{D}(g)$ has the same splitting type as $\mathcal{D}(f)$.
- (b) If $\mathcal{A}(g)$ is not free, then the splitting type of $\mathcal{D}(g)$ is $(a - s, b + s)$ for some positive integer s .

Proof. Set $d = \deg f$. By Theorem 7.2, since $\mathcal{A}(f)$ is free we get $c_2(\mathcal{D}(f)) = ab$. Since the arrangements have the same incidence lattice, Equation (7.1) gives $c_2(\mathcal{D}(g)) = c_2(\mathcal{D}(f))$. Combining, we obtain $c_2(\mathcal{D}(g)) = ab$.

Since f and g have the same degree, the sum of the integers in the splitting type for $\mathcal{D}(f)$ must be equal to the sum for $\mathcal{D}(g)$, i.e. the splitting type for $\mathcal{D}(g)$ is $(a - s, b + s)$ for

some integer s , where $a - s \leq b + s$. Combined with Theorem 7.2, and using the fact that $a + b + 1 = d$, we obtain

$$0 \leq c_2(\mathcal{D}(g)) - (a - s)(b + s) = ab - (a - s)(b + s) = a(d - 1 - a) - (a - s)(d - 1 - a + s).$$

Since the function $h(t) = t(d - 1 - t)$ is strictly increasing on the interval $(-\infty, \frac{d-1}{2}]$, and since both a and $a - s$ lie in this interval, we conclude that $s \geq 0$ and that $c_2(\mathcal{D}(g)) - (a - s)(b + s) = 0$ if and only if $s = 0$. Hence Theorem 7.2 gives that $\mathcal{D}(g)$ is free if and only if $s = 0$ and that $s > 0$ otherwise. \square

As an immediate consequence we get:

Corollary 7.4. *If the splitting type of a line arrangement is a combinatorial property for free arrangements, then Terao's conjecture is true for line arrangements.*

The analogous question is of interest also for non-free arrangements. Thus we propose the following question:

Question 7.5. *Is the splitting type a combinatorial invariant for arbitrary arrangements?*

Using a Lefschetz-like property, we give a statement that is equivalent to Terao's conjecture.

Proposition 7.6. *The following two conditions are equivalent:*

- (a) *Terao's conjecture is true.*
- (b) *If $\mathcal{A}(f)$ is any free line arrangement with splitting type (a, b) , then, for every line arrangement $\mathcal{A}(g)$ with the same incidence lattice as $\mathcal{A}(f)$, the multiplication map*

$$[R/J]_{b-2} \xrightarrow{\times L^2} [R/J]_b$$

is surjective, where $J = (\ell_1^b, \dots, \ell_{a+b+1}^b, L_1^b, \dots, L_{b-a}^b)$ with $g = \ell_1 \cdots \ell_{a+b+1}$ and general linear forms $L, L_1, \dots, L_{b-a} \in R$.

Proof. Let $\mathcal{A}(f)$ be a free line arrangement with splitting type (a, b) , and let $\mathcal{A}(g)$ be a line arrangement with the same incidence lattice as $\mathcal{A}(f)$. By Proposition 7.3, the splitting type of $\mathcal{A}(g)$ is $(a - s, b + s)$ for some integer $s \geq 0$. Let $L, L_1, \dots, L_{b-a} \in R$ be general linear forms, and set $h = L_1 \cdots L_{b-a}$. Proposition 5.9 gives that $\mathcal{A}(gh)$ has splitting type $(b - s, b + s)$. Denote by Z the set of points in \mathbb{P}^2 that is dual to $\mathcal{A}(gh)$, and let $P \in \mathbb{P}^2$ be the point that is dual to L . Thus, Z has multiplicity index $m_Z = b - s$ by Theorem 5.2. The cokernel of the multiplication map

$$[R/J]_{b-2} \xrightarrow{\times L^2} [R/J]_b$$

is $[R/(J, L^2)]_b$. By Theorem 6.2, this is isomorphic to $[I_{Z+(b-1)P}]_b$. It follows that the above map is surjective if and only if $m_Z = b$, that is, $s = 0$, which means that $\mathcal{A}(g)$ has the same splitting type as $\mathcal{A}(f)$. By Proposition 7.3, the latter is equivalent to $\mathcal{A}(g)$ being free, which concludes the argument. \square

Similar arguments give a sufficient condition.

Corollary 7.7. *Consider the following condition*

(*) Let $f = \ell'_1 \cdots \ell'_{2k+1}$ and $g = \ell_1 \cdots \ell_{2k+1}$ be products of $2k+1$ linear forms in R , and let $L \in R$ be a general linear form. Assume that the multiplication map

$$[R/I]_{k-2} \xrightarrow{\times L^2} [R/I]_k$$

is surjective, where $I = (\ell_1^k, \dots, \ell_{2k+1}^k)$.

If the line arrangements $\mathcal{A}(f)$ and $\mathcal{A}(g)$ have the same incidence lattices, then the multiplication map

$$[R/J]_{k-2} \xrightarrow{\times L^2} [R/J]_k$$

is also surjective, where $J = (\ell_1^k, \dots, \ell_{2k+1}^k)$.

If Condition (*) is true for any two sets of $2k+1$ linear forms, then Terao's conjecture is true.

Proof. Adopt the notation of the proof of Proposition 7.6. In particular, let $\mathcal{A}(f)$ and $\mathcal{A}(g)$ be two line arrangements with the same incidence lattice, where $\mathcal{A}(f)$ is free with splitting type (a, b) . Let $\ell'_1, \dots, \ell'_{a+b+1}$ be linear forms such that $f = \ell'_1 \cdots \ell'_{a+b+1}$. We will use Condition (*) by considering the ideal $I = (\ell_1^b, \dots, \ell_{a+b+1}^b, L_1^b, \dots, L_{b-a}^b)$. Indeed, the arrangement $\mathcal{A}(fh)$ has splitting type (b, b) . Hence the multiplication map

$$[R/I]_{b-2} \xrightarrow{\times L^2} [R/I]_b$$

is surjective. Since L_1, \dots, L_{b-a} are general linear forms, the arrangements $\mathcal{A}(fh)$ and $\mathcal{A}(gh)$ also have the same incidence lattice. Therefore, Condition (*) gives that the multiplication map

$$[R/J]_{b-2} \xrightarrow{\times L^2} [R/J]_b$$

is surjective, where $J = (\ell_1^b, \dots, \ell_{a+b+1}^b, L_1^b, \dots, L_{b-a}^b)$. As above, it follows that $\mathcal{A}(g)$ must be a free arrangement, as desired. \square

Remark 7.8. (i) In [DIV] the authors conjecture that the above Condition (*) is always satisfied if one replaces surjectivity of the multiplication maps by maximal rank. Moreover, they claim that this modification of Condition (*) is equivalent to Terao's conjecture.

(ii) We have seen in Example 6.4 that injectivity of the multiplication map is not enough to draw a conclusion on the splitting type. One needs surjectivity as stated in Condition (*). However, it is not clear (to us) whether Condition (*) is in fact equivalent to Terao's conjecture.

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